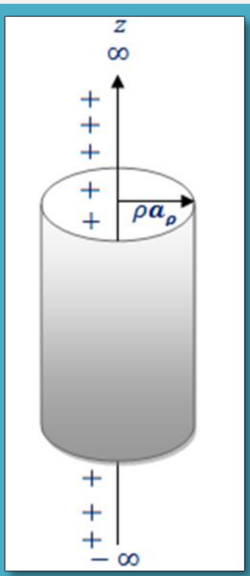
ISBN 978-969-23460-0-9

Electromagnetic Field Theory



First Edition

2020

Dr. Gulzar Ahmad
Associate Professor
UET Peshawar

ISBN 978-969-23460-0-9

Table of Contents

Preface

Electromagnetic Field Theory is a fundamental and probably one of the most complicated subjects for students of undergraduate level. When I was pursuing my studies towards second master degree in Electrical Engineering from George Washington University USA, I learnt the subject with intention to teach it to my undergraduate students in a way to make it a user friendly subject for them. During my studies the two books, *Electromagnetic Field Theory, By Hozorgu & Guru and Engineering Electromagnetics, 5th Edition by William H. Hayt, J*r. helped me a lot and I have used these books as reference books with an objective that it will help my undergraduate students. This book is intended to be easy and bringing the readers the important information regarding some basic and fundamental topics of Electromagnetic Field Theory. Important theoretical and mathematical results are given with the accompanying lengthy proofs, which I think is the main characteristic of the book. Solved numerical problems have been added to give the students the confidence in understanding the material presented. This book covers the topics of basic Electromagnetic Field Theory with the objective of learning and motivation. Easy explanation of topics and plenty of solved relevant examples is the principal features of this book. Practice problems have not been included in the first edition and I intend to include it in the next edition.

The book is designed primarily for junior-level undergraduate university students and is one semester course. It includes 8 chapters on Vector Quantities, Force and Electric Field Intensity, Electric Flux & Electric Flux Density, Energy and Voltage, Magnetic Flux and Fields, Force and Torque, Maxwell's Equations and Electromagnetic Waves. This is my second book as the first book titled; Basic Electrical Engineering for the undergraduate level students of Electrical Engineering was published in 2019. I know there will be lots of errors in this book, but the feedback of the students for the rectification of the errors will be appreciated and will definitely enhance its quality.

Dr. Gulzar Ahmad
Associate Professor
Department of Electrical Engineering
University of Engineering & Technology Peshawar, Pakistan

Chapter 1

Vector Quantities

1-1 Vectors

Those quantities that can be described by magnitude as well as direction are known as vectors. For example force, electric field intensity and magnetic field intensity. Mathematically a vector is represented as

$$\boldsymbol{A} = A\boldsymbol{a} \tag{1.1}$$

Where A represents the magnitude of this vector and $\boldsymbol{a}$ is a unit vector in the direction of $\boldsymbol{A}$. We already know that magnitude of a unit vector is 1.The bold letters represent vectors. The following equation defines a unit vector in the direction of vector $\boldsymbol{A}$.

$$\boldsymbol{a} = \frac{\boldsymbol{A}}{A} \tag{1.2}$$

The graphical representation of a vector is given in Figure 1-1. The magnitude of this vector is given by the length of the arrow and the direction of the vector is given by the direction of arrow.

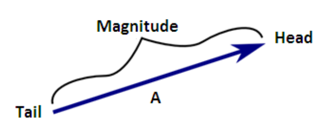


Figure 1-1: Representation of a Vector

1-2 Addition of Vectors

Vectors are added with the help of head to tail rule. The addition of vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ results in a new vector, as shown in Figure 1-2. Vectors obey Commutative Law in case of addition. Mathematically

$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{C} \tag{1.3}$$

$$B + A = C \tag{1.4}$$

So

$$A + B = B + A \tag{1.5}$$

(Commutative Law)

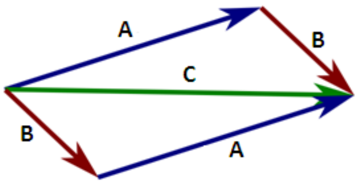


Figure 1- 2: Vectors Addition

1-3 Scalar Product

This product is also known as dot product. It results in a scalar quantity and obeys commutative law. Consider two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ as shown in Figure 1-3. The angle between these two vectors is θ.

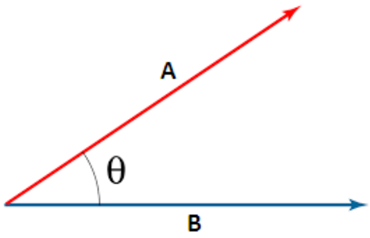


Figure 1-3: Scalar Product

The Scalar product is given by

$$\boldsymbol{A}.\boldsymbol{B} = \boldsymbol{B}.\boldsymbol{A} = AB\cos\theta \tag{1.6}$$

(Commutative Law)

Consider two vectors equal in magnitude and in the same direction as shown in Figure 1- 4.

$$\boldsymbol{A}.\boldsymbol{A} = A^2 \tag{1.7}$$

So the magnitude of a vector can be calculated with the help of the following equation.

$$A = \sqrt{\boldsymbol{A}.\boldsymbol{A}} \tag{1.8}$$

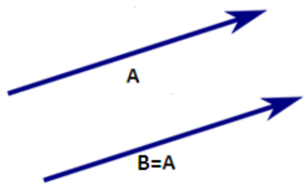


Figure 1-4: Scalar Product of Two Parallel Vectors

1-4 Vector Product

This product is also known as cross product. It results in a vector quantity and does not obey commutative law. Consider two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ in the plane of the paper as shown in Figure 1-5.

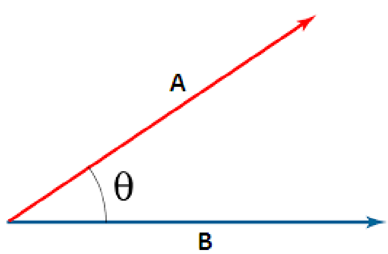


Figure 1-5: Two Vectors

The vector product is given by

$$\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B} = AB \sin\theta \, \boldsymbol{a}_n \tag{1.9}$$

Where $\boldsymbol{a}_n$ is a unit vector normal to the plane containing vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. The direction of the resultant vector can be found using right hand rule. We place our right hand with thumb normal to the fingers along the first vector $\boldsymbol{A}$ such that if we move it towards the second vector $\boldsymbol{B}$ the front side of the right hand faces vector $\boldsymbol{B}$. In this case the thumb indicates the direction of the resultant vector $\boldsymbol{C}$. Note that the cross product doesn't obey commutative law.

$$\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A} \qquad (1.10)$$

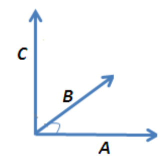


Figure 1-6: Vector Product

1-5 Rectangular Coordinate System

There are three coordinates X, Y and Z in the Rectangular Coordinate System and they are normal to one another as shown in Figure 1-7.

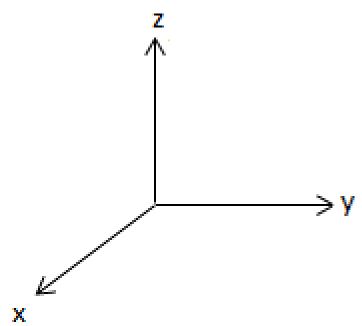


Figure 1-7: Coordinates of Rectangular Coordinate System

There are three planes in the rectangular coordinate system that is $x = 0$ Plane, $y = 0$ Plane and $z = 0$ Plane as illustrated in Figure 1-8. We can find out the location of a point $P(x, y, z)$ in the three dimensional space, if x, y and z coordinates of the point are known. The x coordinate of the point $P(x, y, z)$ is parallel to $x - axis$, y coordinate is parallel to $y - axis$, while the z coordinate of the point is parallel to $z - axis$. If a point is in the $z = 0$ Plane, then its z coordinate will be zero, If a point is in the $x = 0$ Plane, then its x coordinate will be zero and If a point is in the $y = 0$ Plane, then its y coordinate will be zero.

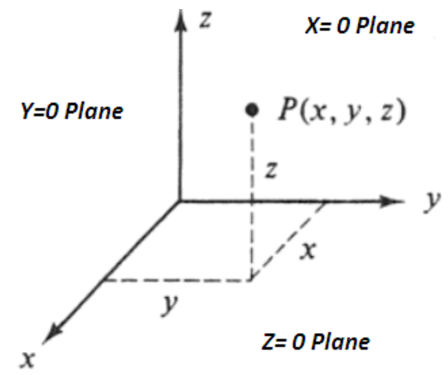


Figure 1-8: Planes in the Rectangular Coordinate system

The three unit vectors $\boldsymbol{a}_x$, $\boldsymbol{a}_y$ and $\boldsymbol{a}_z$ are along x, y and z axis respectively and they are normal to one another as shown in Figure 1-9.

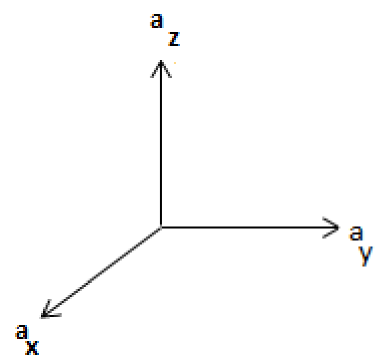


Figure 1-9: Unit Vectors of the Rectangular Coordinate system

We know that $\boldsymbol{a}_x.\boldsymbol{a}_x = \boldsymbol{a}_y.\boldsymbol{a}_y = \boldsymbol{a}_z.\boldsymbol{a}_z = 1$ and $\boldsymbol{a}_x.\boldsymbol{a}_y = \boldsymbol{a}_y.\boldsymbol{a}_z = \boldsymbol{a}_z.\boldsymbol{a}_x = 0$. Let us review that vector obeys commutative law in case of scalar product. It means that if we change the order of the two vectors, the results does not change. Similarly $\boldsymbol{a}_x \times \boldsymbol{a}_x = \boldsymbol{a}_y \times \boldsymbol{a}_y = \boldsymbol{a}_z \times \boldsymbol{a}_x = 0$. The cross product of the unit vectors is elaborated in Table 1-1. It is evident from the table that cross product does not obey commutative law.

Table 1-1: Cross Product of Unit Vectors

$\boldsymbol{a}_x \times \boldsymbol{a}_y = \boldsymbol{a}_z$	$\boldsymbol{a}_y \times \boldsymbol{a}_x = -\boldsymbol{a}_z$
$\boldsymbol{a}_y \times \boldsymbol{a}_z = \boldsymbol{a}_x$	$\boldsymbol{a}_z \times \boldsymbol{a}_y = -\boldsymbol{a}_x$
$\boldsymbol{a}_z \times \boldsymbol{a}_x = \boldsymbol{a}_y$	$\boldsymbol{a}_x \times \boldsymbol{a}_z = -\boldsymbol{a}_y$

A three dimensional vector in rectangular coordinate system has three components. The first component is along x-axis, the second component is along y-axis and the third component is along z-axis as given in equation 1.11.

$$\boldsymbol{A} = A_x\boldsymbol{a_x} + A_y\boldsymbol{a_y} + A_z\boldsymbol{a_z} \tag{1.11}$$

1-5-1 Addition of Vectors

The addition of two or more than two vectors results in a new vector. Vector addition obeys commutative law. Consider the addition of the following two vectors.

$$\boldsymbol{A} = A_x\boldsymbol{a_x} + A_y\boldsymbol{a_y} + A_z\boldsymbol{a_z} \tag{1.12}$$

$$\boldsymbol{B} = B_x\boldsymbol{a_x} + B_y\boldsymbol{a_y} + B_z\boldsymbol{a_z} \tag{1.13}$$

$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A} = (A_x + B_x)\boldsymbol{a_x} + \left(A_y + B_y\right)\boldsymbol{a_y} + (A_z + B_z)\boldsymbol{a_z} \tag{1.14}$$

1-5-2 Scalar Product of Vectors

The scalar product of two vectors results in a scalar. It obeys commutative law. Consider the scalar product of the two vectors $\boldsymbol{A}$ & $\boldsymbol{B}$.

$$\boldsymbol{A}.\boldsymbol{B} = \boldsymbol{B}.\boldsymbol{A} = (A_xB_x) + \left(A_yB_y\right) + (A_zB_z) \tag{1.15}$$

Consider two vectors which are equal in magnitude and in the same direction, then

$$\boldsymbol{A}.\boldsymbol{A} = (A_xA_x) + \left(A_yA_y\right) + (A_zA_z)$$

$$\boldsymbol{A}.\boldsymbol{A} = (A_x)^{\mathbf{2}} + \left(A_y\right)^{\mathbf{2}} + (A_z)^{\mathbf{2}}$$

The magnitude of a vector in RCS can be calculated as

$$\boldsymbol{A} = \sqrt{\boldsymbol{A}.\boldsymbol{A}} = \sqrt{(A_x)^{\mathbf{2}} + \left(A_y\right)^{\mathbf{2}} + (A_z)^{\mathbf{2}}} \tag{1.16}$$

1-5-3 Cross Product of Vectors

The cross product of two vectors results in a vector quantity. It does not obey commutative law. Consider the cross product of the two vectors $\boldsymbol{A}$ & $\boldsymbol{B}$.

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

While

$$\boldsymbol{B} \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix}$$

In rectangular coordinate system, we assume that this three dimensional universe is in the form of a rectangular box as shown in Figure 1-10.

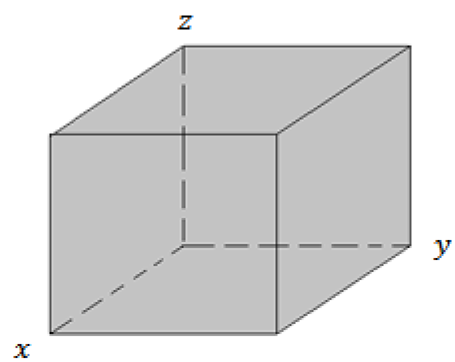


Figure 1-10: Rectangular Box

Limits for the three coordinates are

$$-\infty \leq x \leq \infty$$

$$-\infty \leq y \leq \infty$$

$$-\infty \leq z \leq \infty$$

Example 1-1:

Consider the vectors in Rectangular coordinate system. (i) add these two vectors (ii) compute the scalar product (iii) compute the vector product (iv) Find the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$.

$$\boldsymbol{A} = 3\boldsymbol{a}_x + 2\boldsymbol{a}_y + 2\boldsymbol{a}_z$$

$$\boldsymbol{B} = \boldsymbol{a}_x + 3\boldsymbol{a}_y + 4\boldsymbol{a}_z$$

Solution

(i) $$\boldsymbol{A} + \boldsymbol{B} = 4\boldsymbol{a}_x + 5\boldsymbol{a}_y + 6\boldsymbol{a}_z$$

(ii) $$\boldsymbol{A}.\boldsymbol{B} = 3 \times 1 + 2 \times 3 + 2 \times 4 = 17$$

(iii)

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 3 & 2 & 2 \\ 1 & 3 & 4 \end{vmatrix}$$

$$\boldsymbol{A} \times \boldsymbol{B} = 2\boldsymbol{a}_x - 10\boldsymbol{a}_y + 7\boldsymbol{a}_z$$

(iv) $$A = \sqrt{9 + 4 + 4} = 4.12$$

$$B = \sqrt{1 + 9 + 16} = 5.09$$

$$\boldsymbol{A}.\boldsymbol{B} = AB \cos\theta$$

$$\theta = \cos^{-1}(\frac{\boldsymbol{A}.\boldsymbol{B}}{AB})$$

$$\theta = \cos^{-1}(\frac{17}{20.97})$$

$$\theta = \cos^{-1}(0.81\,) = 35.9^0$$

Example 1-2:

Consider the vectors in rectangular coordinate system. Find the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$ with the help of vector product.

$$\boldsymbol{A} = 3\boldsymbol{a}_x + 4\boldsymbol{a}_y + 0\boldsymbol{a}_z$$

$$\boldsymbol{B} = \mathbf{4}\boldsymbol{a}_x + 3\boldsymbol{a}_y + \boldsymbol{a}_z$$

Solution

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ 3 & 4 & 0 \\ 4 & 3 & 1 \end{vmatrix}$$

$$\boldsymbol{A} \times \boldsymbol{B} = 4\boldsymbol{a_x} - 3\boldsymbol{a_y} - 7\boldsymbol{a_z}$$

$$|\boldsymbol{A} \times \boldsymbol{B}| = \sqrt{16 + 9 + 49} = 8.6$$

$$A = \sqrt{9 + 16} = 5$$

$$B = \sqrt{16 + 9 + 1} = 5.09$$

$$\theta = \sin^{-1}(\frac{|A \times B|}{AB})$$

$$\theta = \sin^{-1}(\frac{8.6}{5 \times 5.09}) = 19.7^0$$

1.6 Cylindrical Coordinate System

We can find out the location of a point P(x, y, z) in the three dimensional spaces with the help of ρ, Ø and z coordinates of the point as well. The location of the point P has been elaborated in Figure 1-11.

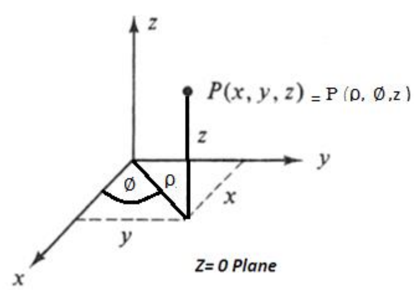


Figure 1-11: Point in Cylindrical Coordinate system

ρ is the radial line in the $z = 0$ plane, which makes an angle of Ø with respect to $x - axis$. So we draw a line ρ in the $z = 0$ plane such that it makes an angle of Ø with

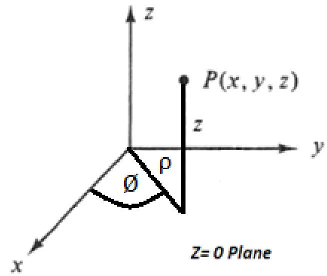


Figure 1-12: Point in Cylindrical Coordinate System

respect to $x - axis$ and then another line z that is perpendicular to ρ and we get the point P as shown in Figure 1-12.
If $\emptyset$ is zero, then radial line ρ will be along $x - axis$ and if $\emptyset = 90$, then radial line ρ will be along $y - axis$. Consider the triangle shown in Figure 1-13.

$$x = \rho \cos \emptyset$$

$$y = \rho \sin \emptyset$$

$$z = z$$

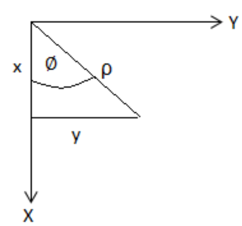


Figure 1-13: Transformation of a point

We apply Pythagoras Theorem to the right angle triangle

$$\rho = \sqrt{x^2 + y^2}$$

$$\emptyset = \tan^{-1} \frac{y}{x}$$

$$z = z$$

There are three unit vectors $\boldsymbol{a}_\rho$, $\boldsymbol{a}_\emptyset$ and $\boldsymbol{a}_z$ in the cylindrical coordinate system as shown in Figure 1-14.These three unit vectors are normal to one another. The unit vector $\boldsymbol{a}_\rho$ is in the $z = 0$ plane such that it makes an angle of $\emptyset$ with respect to x-axis.

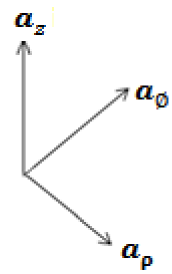


Figure 1-14: Unit Vectors of Cylindrical Coordinate System

The unit vector $\boldsymbol{a}_\emptyset$ is in the $z = 0$ plane such that it makes an angle of $90 + \emptyset$ with respect to x-axis and the unit vector $\boldsymbol{a}_z$ is normal to the plane containing $\boldsymbol{a}_\rho$ and $\boldsymbol{a}_\emptyset$. Let us consider a few examples. If $\emptyset = 0$, then $\boldsymbol{a}_\rho$ will be along x-axis, $\boldsymbol{a}_\emptyset$ will be along $y - axis$ and $\boldsymbol{a}_z$ will be along $z - axis$ as shown in Figure 1-15.

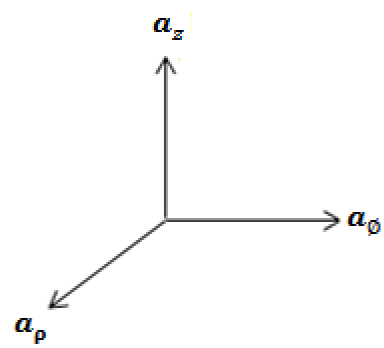


Figure 1-15: Unit Vectors when $\emptyset = 0$

If $\emptyset = 90$, then $\boldsymbol{a}_\rho$ will be along $y - axis$, $\boldsymbol{a}_\emptyset$ will be along negative $x - axis$ and $\boldsymbol{a}_z$ will be along $z - axis$ as shown in Figure 16.

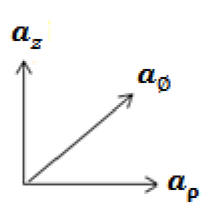


Figure 1-16: Unit Vectors when $\emptyset = 90$

We know that $\boldsymbol{a}_\rho.\boldsymbol{a}_\rho = \boldsymbol{a}_\emptyset.\boldsymbol{a}_\emptyset = \boldsymbol{a}_z.\boldsymbol{a}_z = 1$ and $\boldsymbol{a}_\rho.\boldsymbol{a}_\emptyset = \boldsymbol{a}_\emptyset.\boldsymbol{a}_z = \boldsymbol{a}_z.\boldsymbol{a}_\rho = 0$. Let us review that vector obeys commutative law in case of scalar product. It means that if we change the order of the two vectors, the results does not change. Similarly $\boldsymbol{a}_{\boldsymbol{\rho}} \times \boldsymbol{a}_{\boldsymbol{\rho}} = \boldsymbol{a}_{\emptyset} \times \boldsymbol{a}_{\emptyset} = \boldsymbol{a}_{\boldsymbol{z}} \times \boldsymbol{a}_{\boldsymbol{z}} = 0$. The cross product of the unit vectors is elaborated in Table 1-2. It is evident from the table that cross product does not obey commutative law.

Table 1-2: Cross Product of Unit Vectors

$\boldsymbol{a}_{\boldsymbol{\rho}} \times \boldsymbol{a}_{\emptyset} = \boldsymbol{a}_{\boldsymbol{z}}$	$\boldsymbol{a}_{\emptyset} \times \boldsymbol{a}_{\boldsymbol{\rho}} = -\boldsymbol{a}_{\boldsymbol{z}}$
$\boldsymbol{a}_{\emptyset} \times \boldsymbol{a}_{\boldsymbol{z}} = \boldsymbol{a}_{\boldsymbol{\rho}}$	$\boldsymbol{a}_{\boldsymbol{z}} \times \boldsymbol{a}_{\emptyset} = -\boldsymbol{a}_{\boldsymbol{\rho}}$
$\boldsymbol{a}_{\boldsymbol{z}} \times \boldsymbol{a}_{\boldsymbol{\rho}} = \boldsymbol{a}_{\emptyset}$	$\boldsymbol{a}_{\boldsymbol{\rho}} \times \boldsymbol{a}_{\boldsymbol{z}} = -\boldsymbol{a}_{\emptyset}$

A three dimensional vector in cylindrical coordinate system has three components. The first component is along the unit vector $\boldsymbol{a}_{\boldsymbol{\rho}}$, the second component is along the unit vector $\boldsymbol{a}_{\emptyset}$ and the third component is along z-axis as given in equation 1.17.

$$\boldsymbol{A} = A_{\rho}\boldsymbol{a}_{\boldsymbol{\rho}} + A_{\emptyset}\boldsymbol{a}_{\emptyset} + A_{z}\boldsymbol{a}_{\boldsymbol{z}} \tag{1.17}$$

1-6-1 Addition of Vectors

The addition of two or more than two vectors results in a new vector. Vector addition obeys commutative law. Consider the addition of the following two vectors. The angle $\emptyset$ should be the same for these two vectors $\boldsymbol{A}$ & $\boldsymbol{B}$. This condition needs to be satisfied for the addition of these two vectors, otherwise we have to transform these two vectors from the cylindrical coordinate system into rectangular coordinate system for the sake of addition.

$$\boldsymbol{A} = A_{\rho}\boldsymbol{a}_{\boldsymbol{\rho}} + A_{\emptyset}\boldsymbol{a}_{\emptyset} + A_{z}\boldsymbol{a}_{\boldsymbol{z}} \tag{1.18}$$

$$\boldsymbol{B} = B_{\rho}\boldsymbol{a}_{\boldsymbol{\rho}} + B_{\emptyset}\boldsymbol{a}_{\emptyset} + B_{z}\boldsymbol{a}_{\boldsymbol{z}} \tag{1.19}$$

$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A} = \left(A_{\rho} + B_{\rho}\right)\boldsymbol{a}_{\boldsymbol{\rho}} + (A_{\emptyset} + B_{\emptyset})\boldsymbol{a}_{\emptyset} + (A_{z} + B_{z})\boldsymbol{a}_{\boldsymbol{z}} \tag{1.20}$$

1-6-2 Scalar Product of Vectors

The scalar product of two vectors results in a scalar. It obeys commutative law. Consider the scalar product of the two vectors $\boldsymbol{A}$ & $\boldsymbol{B}$.The angle $\emptyset$ should be the same for these two vectors $\boldsymbol{A}$ & $\boldsymbol{B}$. This condition needs to be satisfied for the scalar product of these two vectors, otherwise we have to transform these two vectors from the cylindrical coordinate system into rectangular coordinate system for the sake of scalar product.

$$\boldsymbol{A}.\boldsymbol{B} = \boldsymbol{B}.\boldsymbol{A} = \left(A_\rho B_\rho\right) + (A_\emptyset B_\emptyset) + (A_z B_z) \qquad (1.21)$$

Consider two vectors which are equal in magnitude and in the same direction, then

$$\boldsymbol{A}.\boldsymbol{A} = \left(A_\rho A_\rho\right) + (A_\emptyset A_\emptyset) + (A_z A_z)$$

$$\boldsymbol{A}.\boldsymbol{A} = \left(A_\rho\right)^{\mathbf{2}} + (A_\emptyset)^{\mathbf{2}} + (A_z)^{\mathbf{2}}$$

The magnitude of a vector in CCS can be calculated as

$$\boldsymbol{A} = \sqrt{\boldsymbol{A}.\boldsymbol{A}} = \sqrt{\left(A_\rho\right)^{\mathbf{2}} + (A_\emptyset)^{\mathbf{2}} + (A_z)^{\mathbf{2}}} \qquad (1.22)$$

1-6-3 Cross Product of Vectors

The cross product of two vectors results in a vector quantity. It does not obey commutative law. Consider the cross product of the two vectors $\boldsymbol{A}$ & $\boldsymbol{B}$.The angle $\emptyset$ should be the same for these two vectors $\boldsymbol{A}$ & $\boldsymbol{B}$. This condition needs to be satisfied for the cross product of these two vectors, otherwise we have to transform these two vectors from the cylindrical coordinate system into rectangular coordinate system for the sake of vector product.

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{a_\rho} & \boldsymbol{a_\emptyset} & \boldsymbol{a_z} \\ A_\rho & A_\emptyset & A_z \\ B_\rho & B_\emptyset & B_z \end{vmatrix}$$

While

$$\boldsymbol{B} \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{a_\rho} & \boldsymbol{a_\emptyset} & \boldsymbol{a_z} \\ B_\rho & B_\emptyset & B_z \\ A_\rho & A_\emptyset & A_z \end{vmatrix}$$

In cylindrical coordinate system, we assume that this three dimensional universe is in the form of a right circular cylinder as shown in Figure 1-17. Radius of the cylinder is ρ that is always normal to $z - axis$.

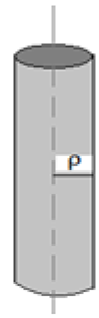


Figure 1-17: Cylinder of radius ρ

Limits for the three coordinates are

$$0 \leq \rho \leq \infty$$

$$0 \leq \emptyset \leq 2\pi$$

$$-\infty \leq z \leq \infty$$

Example 1-3:

Find distance between *A* and *B*.

$$A(8,6,0) \qquad ; \qquad B(5, 36.8^0, 0)$$

Solution

We transform B to rectangular coordinate system

$$x = 5\cos 36.8 = 4$$

$$y = 5\sin 36.8 = 3$$

$$z = 0$$

So $B(5, 36.8^0, 0)$ in rectangular coordinate system is given by

$$B(4,3,0)$$

Distance between A and B is given by

$$d = \sqrt{(8-4)^2 + (6-3)^2}$$

$$d = 5$$

1-7 Spherical Coordinate System

We found the location of the point P(x, y, z) in the three dimensional spaces with the help of ρ, ∅ and z coordinates of the point in the cylindrical coordinate system. The location of the point P was elaborated in Figure 1-18. We replace ρ by $r \sin\theta$ and z by $r\cos\theta$ in Figure 1-18 and draw the same point P with the help of r, θ and ∅. It can be seen from Figure 1-19 that

$$\rho = r\sin\theta$$

$$z = r\cos\theta$$

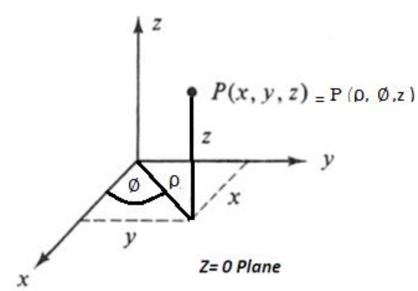


Figure 1-18: Point in Cylindrical Coordinate system

So we draw a line $r\sin\theta$ in the $z = 0$ plane such that it makes an angle ∅ with respect to $x - axis$ and then another line $r\cos\theta$ that is perpendicular to $r\sin\theta$ and we get the point P as shown in Figure 1-19.

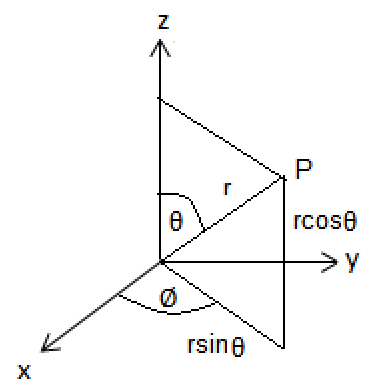


Figure 1-19: Point in Spherical Coordinate System

As

$$x = \rho \cos \emptyset$$

Putting the value of ρ in above equation, we obtain

$$x = r \sin \theta \cos \emptyset \tag{1.23}$$

And as

$$y = \rho \sin \emptyset$$

Putting the value of ρ in above equation, we obtain

$$y = r \sin \theta \sin \emptyset \tag{1.24}$$

$$z = r \cos \theta \tag{1.25}$$

We square and add equations (1.23) and (1.24)

$$x^2 = r^2 \sin^2 \theta \cos^2 \emptyset$$

$$y^2 = r^2 \sin^2 \theta \sin^2 \emptyset$$

$$x^2 + y^2 = r^2 \sin^2 \theta \tag{1.26}$$

Taking the square on both sides of equation 1.25 and adding with equation 1.26, we obtain

$$x^2 + y^2 = r^2 \sin^2 \theta$$

$$z^2 = r^2 \cos^2 \theta$$

$$\overline{}$$

$$r^2 = x^2 + y^2 + z^2$$

Therefore

$$r = \sqrt{x^2 + y^2 + z^2} \qquad (1.27)$$

$$\emptyset = \tan^{-1} \frac{y}{x} \qquad (1.28)$$

$$\theta = \cos^{-1} \frac{z}{r} \qquad (1.29)$$

There are three unit vectors $\boldsymbol{a}_{\mathrm{r}}$, $\boldsymbol{a}_{\emptyset}$ and $\boldsymbol{a}_{\boldsymbol{\theta}}$ in the Spherical coordinate system as shown in Figure 20. These three unit vectors are normal to one another. The unit vector $\boldsymbol{a}_{\mathbf{r}}$ is along the radial line r and it makes an angle of θ with respect to $z - axis$. The unit vector $\boldsymbol{a}_{\emptyset}$ is in the $z = 0$ plane such that it makes an angle of $90 + \emptyset$ with respect to $x - axis$ and the unit vector $\boldsymbol{a}_{\boldsymbol{\theta}}$ is normal to the plane containing $\boldsymbol{a}_{\mathbf{r}}$ and $\boldsymbol{a}_{\emptyset}$. Let us consider a few examples.

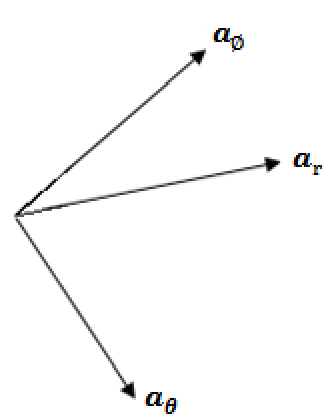


Figure 1-20: Unit Vectors of Spherical Coordinate System

If $\emptyset = 0$, and $\theta = 0$, then $\boldsymbol{a}_{\mathbf{r}}$ will be along z-axis**,** $\boldsymbol{a}_{\emptyset}$ will be along y-axis and $\boldsymbol{a}_{\theta}$ will be along x-axis as shown in Figure 1-21**.**

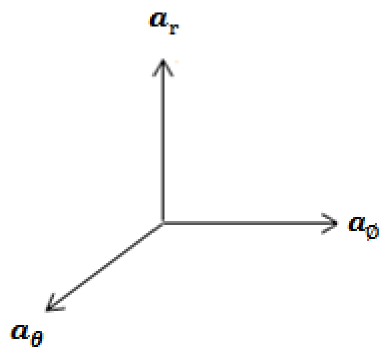


Figure 1-21: Unit Vectors when $\emptyset$ & $\theta = 0$

If $\emptyset = 90$ and $\theta = 0$, then $\boldsymbol{a}_{\mathbf{r}}$ will be along z-axis**,** $\boldsymbol{a}_{\emptyset}$ will be along negative x-axis and $\boldsymbol{a}_{\theta}$ will be along y-axis as shown in Figure 1-22**.**

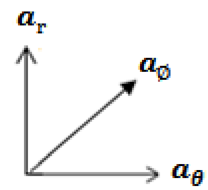


Figure 1-22: Unit Vectors when $\emptyset = 90$ & $\theta = 0$

We know that $\boldsymbol{a}_{\mathbf{r}} . \boldsymbol{a}_{\mathbf{r}} = \boldsymbol{a}_{\emptyset} . \boldsymbol{a}_{\emptyset} = \boldsymbol{a}_{\boldsymbol{\theta}} . \boldsymbol{a}_{\boldsymbol{\theta}} = 1$. Let us review that vector obeys commutative law in case of scalar product.It means that if we change the order of the two vectors, the results do not change. Similarly $\boldsymbol{a}_{\mathbf{r}} \times \boldsymbol{a}_{\mathbf{r}} = \boldsymbol{a}_{\emptyset} \times \boldsymbol{a}_{\emptyset} = \boldsymbol{a}_{\boldsymbol{\theta}} \times \boldsymbol{a}_{\boldsymbol{\theta}} = 0$. The cross product of the unit vectors is elaborated in Table 1-3. It is evident from the table that cross product does not obey commutative law.

Table 1-3: Cross Product of Unit Vectors

$\boldsymbol{a}_{\mathbf{r}} \times \boldsymbol{a}_{\theta} = \boldsymbol{a}_{\boldsymbol{\varphi}}$	$\boldsymbol{a}_{\theta} \times \boldsymbol{a}_{\mathbf{r}} = -\boldsymbol{a}_{\boldsymbol{\varphi}}$
$\boldsymbol{a}_{\theta} \times \boldsymbol{a}_{\boldsymbol{\varphi}} = \boldsymbol{a}_{\mathbf{r}}$	$\boldsymbol{a}_{\boldsymbol{\varphi}} \times \boldsymbol{a}_{\theta} = -\boldsymbol{a}_{\mathbf{r}}$
$\boldsymbol{a}_{\boldsymbol{\varphi}} \times \boldsymbol{a}_{\mathbf{r}} = \boldsymbol{a}_{\theta}$	$\boldsymbol{a}_{\mathbf{r}} \times \boldsymbol{a}_{\boldsymbol{\varphi}} = -\boldsymbol{a}_{\theta}$

A three dimensional vector in spherical coordinate system has three components. The first component is along the unit vector $\boldsymbol{a}_{\mathbf{r}}$, the second component is along the unit vector $\boldsymbol{a}_\emptyset$ and the third component is along the unit vector $\boldsymbol{a}_\theta$ as given in equation 1-30.

$$\boldsymbol{A} = A_{\mathrm{r}}\boldsymbol{a}_{\mathbf{r}} + A_\theta \boldsymbol{a}_{\boldsymbol{\theta}} + A_\emptyset \boldsymbol{a}_\emptyset \qquad (1.30)$$

1-7-1 Addition of Vectors

The addition of two or more than two vectors results in a new vector. Vector addition obeys commutative law. Consider the addition of the following two vectors. The angle ∅ and θ should be the same for these two vectors $\boldsymbol{A}$ & $\boldsymbol{B}$**.** These two conditions need to be satisfied for the addition of these two vectors, otherwise we have to transform these two vectors from the spherical coordinate system into rectangular coordinate system for the sake of addition.

$$\boldsymbol{A} = A_{\mathrm{r}}\boldsymbol{a}_{\mathbf{r}} + A_\theta \boldsymbol{a}_{\boldsymbol{\theta}} + A_\emptyset \boldsymbol{a}_\emptyset \qquad (1.31)$$

$$\boldsymbol{B} = B_{\mathrm{r}}\boldsymbol{a}_{\mathbf{r}} + B_\theta \boldsymbol{a}_{\boldsymbol{\theta}} + B_\emptyset \boldsymbol{a}_\emptyset \qquad (1.32)$$

$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A} = (A_{\mathrm{r}} + B_{\mathrm{r}})\boldsymbol{a}_{\mathbf{r}} + (A_\theta + B_\theta)\boldsymbol{a}_{\boldsymbol{\theta}} + (A_\emptyset + B_\emptyset)\boldsymbol{a}_\emptyset \qquad (1.33)$$

1-7-2 Scalar Product of Vectors

The scalar product of two vectors results in a scalar. It obeys commutative law. Consider the scalar product of the two vectors $\boldsymbol{A}$ & $\boldsymbol{B}$**.** The angle ∅ and θ should be the same for these two vectors $\boldsymbol{A}$ & $\boldsymbol{B}$**.** These two conditions need to be satisfied for the addition of these two vectors, otherwise we have to transform these two vectors from the spherical coordinate system into rectangular coordinate system for the sake of dot product.

$$\boldsymbol{A}.\boldsymbol{B} = \boldsymbol{B}.\boldsymbol{A} = (A_{\mathrm{r}}B_{\mathrm{r}}) + (A_\theta B_\theta) + (A_\emptyset B_\emptyset) \qquad (1.34)$$

Consider two vectors which are equal in magnitude and in the same direction, then

$$\boldsymbol{A}.\boldsymbol{A} = (A_{\mathrm{r}}A_{\mathrm{r}}) + (A_\theta A_\theta) + (A_\emptyset A_\emptyset)$$

$$\boldsymbol{A}.\boldsymbol{A} = (A_{\mathrm{r}})^2 + (A_\theta)^2 + (A_\emptyset)^2$$

The magnitude of a vector in CCS can be calculated as

$$A = \sqrt{\boldsymbol{A}.\boldsymbol{A}} = \sqrt{(A_{\mathrm{r}})^2 + (A_\theta)^2 + (A_\emptyset)^2} \qquad (1.35)$$

1-7-3 Cross Product of Vectors

The cross product of two vectors results in a vector quantity. It does not obey commutative law. Consider the cross product of the two vectors $\boldsymbol{A}$ & $\boldsymbol{B}$**.** The angle $\emptyset$ and θ should be the same for these two vectors $\boldsymbol{A}$ & $\boldsymbol{B}$**.** These two conditions need to be satisfied for the addition of these two vectors, otherwise we have to transform these two vectors from the spherical coordinate system into rectangular coordinate system for the sake of vector product.

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{a}_{\mathbf{r}} & \boldsymbol{a}_\theta & \boldsymbol{a}_\varphi \\ A_{\mathrm{r}} & A_\theta & A_\varphi \\ B_{\mathrm{r}} & B_\theta & B_\varphi \end{vmatrix}$$

While

$$\boldsymbol{B} \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{a}_{r} & \boldsymbol{a}_\theta & \boldsymbol{a}_\varphi \\ B_{\mathrm{r}} & B_\theta & B_\varphi \\ A_{\mathrm{r}} & A_\theta & A_\varphi \end{vmatrix}$$

In spherical coordinate system, we assume that this three dimensional universe is in the form of a sphere of radius r as shown in Figure 1-23.

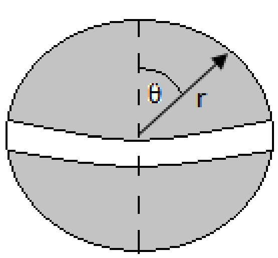


Figure 1-23: Sphere of radius r

Limits for the three coordinates are

$$0 \leq \mathrm{r} \leq \infty$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \emptyset \leq 2\pi$$

Example 1-4:

Find distance between *A* and *B*.

$$A(20, 36.8^0, 0) \qquad ; \qquad B(5, 90^0, 36.8^0)$$

Solution

We transform A to rectangular coordinate system

$$x = 20\cos 36.8 = 16$$

$$y = 20\sin 36.8 = 12$$

$$z = 0$$

So $A(5, 36.8^0, 0)$ in rectangular coordinate system is given by

$$A(16, 12, 0)$$

We transform B to rectangular coordinate system

$$x = 5\sin 90\cos 36.8 = 4$$

$$y = 5\sin 90\sin 36.8 = 3$$

$$z = 5\cos 90 = 0$$

So $B(5, 90^0, 36.8^0)$ in rectangular coordinate system is given by

$$B(4, 3, 0)$$

Distance between A and B is given by

$$d = \sqrt{(16-4)^2 + (12-3)^2}$$

$$d = 15$$

1-8 Component of a Vector in a given direction

Consider vector $\boldsymbol{B}$ as shown in Figure 1-24. We want to find out the scalar component of $\boldsymbol{B}$ along the unit vector $\boldsymbol{a}$ and the vector component of $\boldsymbol{B}$ in the direction of $\boldsymbol{a}$.

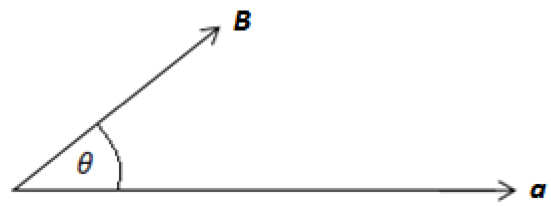


Figure 1-24: Vector $\boldsymbol{B}$ and a unit Vector $\boldsymbol{a}$

We resolve the given vector in to two components and consider the component that is along the unit vector. Consider Figure 1-25, the scalar component of $\boldsymbol{B}$ along the unit vector $\boldsymbol{a}$ is given by

$$\boldsymbol{B}.\boldsymbol{a} = B\cos\theta \tag{1.36}$$

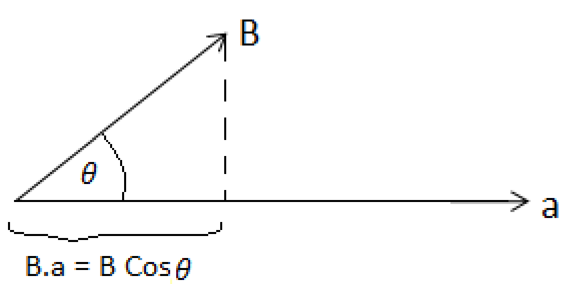


Figure 1-25: Scalar component of $\boldsymbol{B}$

Vector component of $\boldsymbol{B}$ in the direction of the unit vector $\boldsymbol{a}$ is shown in Figure 1-26 and is given by

$$(\boldsymbol{B}.\boldsymbol{a})\boldsymbol{a} = (B\cos\theta)\boldsymbol{a} \tag{1.37}$$

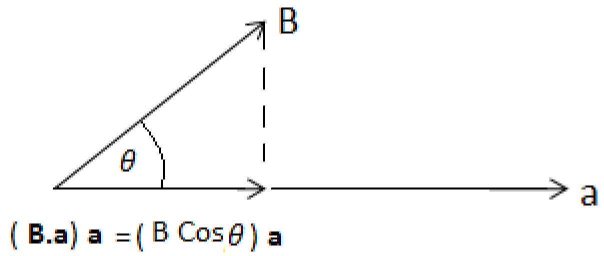


Figure 1- 26: Vector component of $\boldsymbol{B}$

1-9 Transformation of the unit Vectors from RCS to CCS

We want to transform the unit vectors of Rectangular Coordinate System into the unit vectors of Cylindrical Coordinate System. Consider the vectors shown in Figure 1-27. We resolve $\boldsymbol{a}_\rho$ into two components. This unit vector has a component along $x - axis,$ and another component along $y - axis$. There is no component along $z - axis$.

$$\boldsymbol{a}_\rho = \cos\varphi \boldsymbol{a}_x + \sin\varphi \boldsymbol{a}_y + 0\boldsymbol{a}_z \qquad (1.38)$$

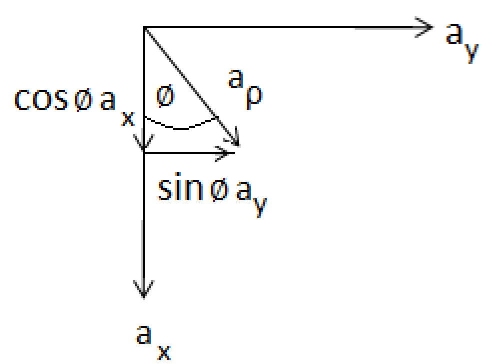


Figure 1-27: Components of $\boldsymbol{a}_\rho$

Consider the vectors shown in Figure 1-28. We resolve $\boldsymbol{a}_\emptyset$ into two components. This unit vector $\boldsymbol{a}_\emptyset$ has a component along negative $x - axis$ and another component along $y - axis$. There is no component along $z - axis$.

$$\boldsymbol{a}_\varphi = -\sin\varphi \boldsymbol{a}_x + \cos\varphi \boldsymbol{a}_y + 0\boldsymbol{a}_z \qquad (1.39)$$

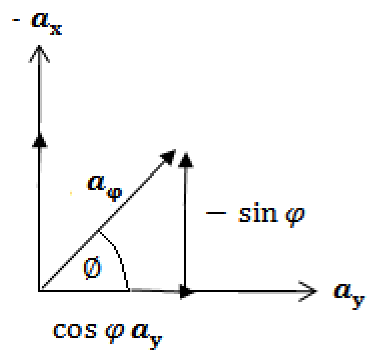


Figure 1-28: Components of $\boldsymbol{a}_\emptyset$

We construct the table of the Scalar product of the unit vectors from equation 1.38 as;

Table 1-4: Scalar Product

$\boldsymbol{a}_\rho \,.\, \boldsymbol{a}_x = \cos\varphi$
$\boldsymbol{a}_\rho \,.\, \boldsymbol{a}_y = \sin\varphi$
$\boldsymbol{a}_\rho \,.\, \boldsymbol{a}_z = 0$

We construct the table of the Scalar product of the unit vectors from equation 1.39 as;

Table 1-5: Scalar Product

$\boldsymbol{a}_\emptyset \,.\, \boldsymbol{a}_x = -\sin\varphi$
$\boldsymbol{a}_\emptyset \,.\, \boldsymbol{a}_y = \cos\varphi$
$\boldsymbol{a}_\emptyset \,.\, \boldsymbol{a}_z = 0$

The unit vector $\boldsymbol{a}_z$ can be written as

$$\boldsymbol{a}_z = 0\,\boldsymbol{a}_x + 0\,\boldsymbol{a}_y + \boldsymbol{a}_z \qquad (1.40)$$

Equations 1.38 -1.40 can be written in the matrices format

$$\begin{bmatrix} \boldsymbol{a}_\rho \\ \boldsymbol{a}_\emptyset \\ \boldsymbol{a}_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_x \\ \boldsymbol{a}_y \\ \boldsymbol{a}_z \end{bmatrix}$$

1-10 Transformation of a Vector from RCS to CCS

We want to transform a vector from Rectangular Coordinate System into Cylindrical Coordinate System. Consider a vector in the Rectangular Coordinate System

$$\boldsymbol{A} = A_x\boldsymbol{a_x} + A_y\boldsymbol{a_y} + A_z\boldsymbol{a_z} \tag{1.41}$$

The scalar component A_ρ is given by

$$A_\rho = \boldsymbol{A} \,.\, \boldsymbol{a_\rho}$$

$$A_\rho = (A_x\boldsymbol{a_x} + A_y\boldsymbol{a_y} + A_z\boldsymbol{a_z}) \,.\, \boldsymbol{a_\rho} \tag{1.42}$$

Equation 1.42 can be simplified with the help of Table 1-4

$$A_\rho = \cos\varphi A_\mathrm{x} + \sin\varphi A_\mathrm{y} + 0A_\mathrm{Z} \tag{1.43}$$

The scalar component $A_\emptyset$ is given by

$$A_\emptyset = \boldsymbol{A} \,.\, \boldsymbol{a}_\emptyset$$

$$A_\emptyset = (A_x\boldsymbol{a_x} + A_y\boldsymbol{a_y} + A_z\boldsymbol{a_z}) \,.\, \boldsymbol{a}_\emptyset \tag{1.44}$$

Equation 44 can be simplified with the help of Table 1-5

$$A_\emptyset = -\sin\varphi\, A_\mathrm{x} + \cos\varphi\, A_\mathrm{y} + 0A_\mathrm{Z} \tag{1.45}$$

The scalar component A_z is given by

$$A_\mathrm{z} = 0\, A_\mathrm{x} + 0\; A_\mathrm{y} + A_\mathrm{Z} \tag{1.46}$$

Equations 1.43, 1.45 and 1.46 can be written in the matrices format as

$$\begin{bmatrix} A_\rho \\ A_\emptyset \\ A_\mathrm{z} \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\mathrm{x} \\ A_\mathrm{y} \\ A_\mathrm{z} \end{bmatrix}$$

We will compute these three cylindrical components of Vector $\boldsymbol{A}$ and then the same vector in cylindrical coordinate system can be written as

$$\boldsymbol{A} = A_\rho\boldsymbol{a_\rho} + A_\emptyset\boldsymbol{a}_\emptyset + A_z\boldsymbol{a_z} \tag{1.47}$$

Example 1.5:

Transform $\boldsymbol{A} = 5\boldsymbol{a}_{\mathbf{x}}$ into Cylindrical Coordinate System at $P(2,\ 53.1^0\ ,3)$

Solution

$$\begin{bmatrix} A_\rho \\ A_\emptyset \\ A_z \end{bmatrix} = \begin{bmatrix} \cos 53.1 & \sin 53.1 & 0 \\ -\sin 53.1 & \cos 53.1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{A} = 4\boldsymbol{a}_{\boldsymbol{\rho}} - 3\boldsymbol{a}_\emptyset + 0\boldsymbol{a}_{\boldsymbol{z}}$$

1-11 Transformation of a Vector from CCS to RCS

We want to transform a vector from Cylindrical Coordinate System into Rectangular Coordinate System. Consider a vector in the Cylindrical Coordinate System

$$\boldsymbol{A} = A_\rho \boldsymbol{a}_{\boldsymbol{\rho}} + A_\emptyset \boldsymbol{a}_\emptyset + A_z \boldsymbol{a}_{\boldsymbol{z}} \qquad (1.48)$$

The scalar component A_x is given by

$$A_x = \boldsymbol{A} \,.\, \boldsymbol{a}_{\mathbf{x}}$$

$$A_x = (A_\rho \boldsymbol{a}_{\boldsymbol{\rho}} + A_\emptyset \boldsymbol{a}_\emptyset + A_z \boldsymbol{a}_{\boldsymbol{z}}) \,.\, \boldsymbol{a}_{\mathbf{x}} \qquad (1.49)$$

Equation 1.49 can be simplified with the help of Table 1-4 and Table 1-5

$$A_x = \cos\varphi\, A_\rho - \sin\varphi\, A_\emptyset + 0A_Z \qquad (1.50)$$

The scalar component A_y is given by

$$A_y = \boldsymbol{A} \,.\, \boldsymbol{a}_{\mathbf{y}}$$

$$A_y = (A_\rho \boldsymbol{a}_{\boldsymbol{\rho}} + A_\emptyset \boldsymbol{a}_\emptyset + A_z \boldsymbol{a}_{\boldsymbol{z}}) \,.\, \boldsymbol{a}_{\mathbf{y}} \qquad (1.51)$$

Equation 1.51 can be simplified with the help of Table 1-4 and Table 1-5

$$A_y = \sin\varphi\, A_\rho + \cos\varphi\, A_\emptyset + 0\, A_Z \qquad (1.52)$$

The scalar component A_z is given by

$$A_z = 0\,A_\rho + \;0\;A_\emptyset + A_Z \tag{1.53}$$

Equations 1.50, 1.52 and 1.53 can be written in the matrices format as

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\emptyset \\ A_z \end{bmatrix}$$

We will compute these three rectangular components of Vector $\boldsymbol{A}$ and then the same vector in rectangular coordinate system can be written as

$$\boldsymbol{A} = A_x\boldsymbol{a_x} + A_y\boldsymbol{a_y} + A_z\boldsymbol{a_z} \tag{1.54}$$

1-12 Transformation of the unit Vectors from RCS to SCS

We want to transform the unit vectors of Rectangular Coordinate System into the unit vectors of Spherical Coordinate System. Consider the vectors shown in Figure 1-29. We resolve $\boldsymbol{a}_\mathrm{r}$ into two components.

$$\boldsymbol{a}_\mathrm{r} = \sin\theta\,\boldsymbol{a}_\rho + \;\cos\theta\,\boldsymbol{a}_\mathrm{z} \tag{1.55}$$

As $$\boldsymbol{a}_\rho = \cos\varphi\boldsymbol{a}_\mathrm{x} + \;\sin\varphi\boldsymbol{a}_\mathrm{y}$$

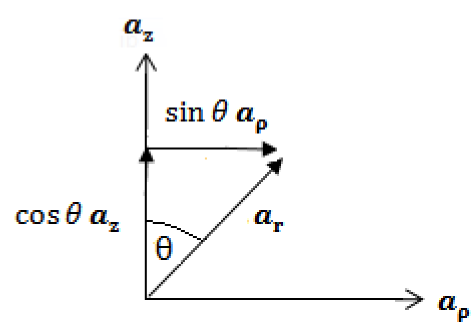


Figure 1-29: Components of $\boldsymbol{a}_\mathrm{r}$

Therefore $$\boldsymbol{a}_\mathrm{r} = \sin\theta\cos\varphi\,\boldsymbol{a}_\mathrm{x} + \sin\theta\sin\varphi\,\boldsymbol{a}_\mathrm{y} + \cos\theta\,\boldsymbol{a}_\mathrm{z} \tag{1.56}$$

We construct the table of the Scalar product of the unit vectors from equation 1.56 as;

Table 1-6: Scalar Product

$\boldsymbol{a}_\mathrm{r} \,.\, \boldsymbol{a}_\mathrm{x} = \sin\theta\cos\varphi$
$\boldsymbol{a}_\mathrm{r} \,.\, \boldsymbol{a}_\mathrm{y} = \sin\theta\sin\varphi$
$\boldsymbol{a}_\mathrm{r} \,.\, \boldsymbol{a}_\mathrm{z} = \cos\theta$

Consider the vectors shown in Figure 1-30. We resolve $\boldsymbol{a}_\theta$ into two components.

$$\boldsymbol{a}_\theta = \sin\alpha\,\boldsymbol{a}_\rho - \cos\alpha\,\boldsymbol{a}_\mathrm{z} \tag{1.57}$$

As $\theta + 90 + \alpha = 180$, therefore $\alpha = 90 - \theta$

$$\boldsymbol{a}_\theta = \cos\theta\,\boldsymbol{a}_\rho - \sin\theta\,\boldsymbol{a}_\mathrm{z} \tag{1.58}$$

Putting the value of $\boldsymbol{a}_\rho$ we obtain

$$\boldsymbol{a}_\theta = \cos\theta\cos\varphi\,\boldsymbol{a}_\mathrm{x} + \cos\theta\sin\varphi\,\boldsymbol{a}_\mathrm{y} - \sin\theta\,\boldsymbol{a}_\mathrm{z} \tag{1.59}$$

We have already resolved $\boldsymbol{a}_\emptyset$ into two components

$$\boldsymbol{a}_\varphi = -\sin\varphi\boldsymbol{a}_\mathrm{x} + \cos\varphi\boldsymbol{a}_\mathrm{y} + 0\boldsymbol{a}_\mathrm{Z} \tag{1.60}$$

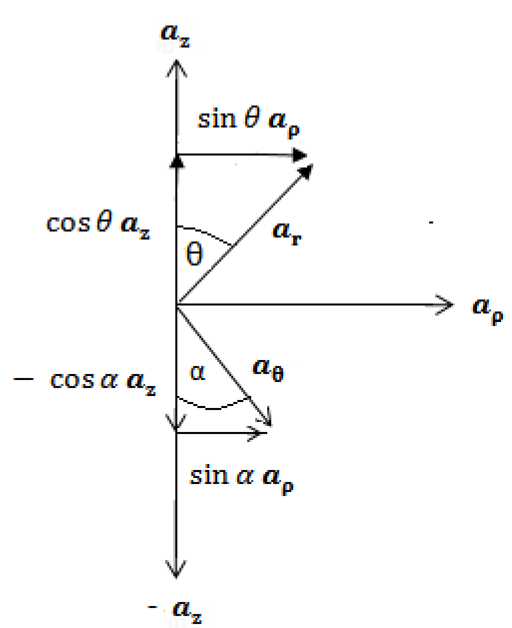


Figure 1-30: Components of $\boldsymbol{a}_\theta$

We construct the table of the Scalar product of the unit vectors from equation 1.59 as;

Table 1.7: Scalar Product

$\boldsymbol{a}_{\boldsymbol{\theta}} \,.\, \boldsymbol{a}_{\mathbf{x}} = \cos\theta\cos\varphi$
$\boldsymbol{a}_{\boldsymbol{\theta}} \,.\, \boldsymbol{a}_{\mathbf{y}} = \cos\theta\sin\varphi$
$\boldsymbol{a}_{\boldsymbol{\theta}} \,.\, \boldsymbol{a}_{\mathbf{z}} = -\sin\theta$

Equations 1.56, 1.59 and 1.60 can be written in the matrices format

$$\begin{bmatrix} \boldsymbol{a}_{\mathbf{r}} \\ \boldsymbol{a}_{\boldsymbol{\theta}} \\ \boldsymbol{a}_{\emptyset} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_{\mathbf{x}} \\ \boldsymbol{a}_{\mathbf{y}} \\ \boldsymbol{a}_{\mathbf{z}} \end{bmatrix}$$

1-13 Transformation of a Vector from RCS to SCS

We want to transform a vector from Rectangular Coordinate System into Spherical Coordinate System. Consider a vector in the Rectangular Coordinate System

$$\boldsymbol{A} = A_x\boldsymbol{a}_{\boldsymbol{x}} + A_y\boldsymbol{a}_{\boldsymbol{y}} + A_z\boldsymbol{a}_{\boldsymbol{z}} \qquad (1.61)$$

The scalar component A_{r} is given by

$$A_{\mathrm{r}} = \boldsymbol{A} \,.\, \boldsymbol{a}_{\mathbf{r}}$$

$$A_{\mathrm{r}} = (A_x\boldsymbol{a}_{\boldsymbol{x}} + A_y\boldsymbol{a}_{\boldsymbol{y}} + A_z\boldsymbol{a}_{\boldsymbol{z}}) \,.\, \boldsymbol{a}_{\mathbf{r}} \qquad (1.62)$$

Equation 1.62 can be simplified with the help of Table 1-6

$$A_{\mathrm{r}} = \sin\theta\cos\varphi A_{\mathrm{x}} + \sin\theta\sin\varphi A_{\mathrm{y}} + \cos\theta\, A_{\mathrm{Z}} \qquad (1.63)$$

The scalar component A_{θ} is given by

$$A_{\theta} = \boldsymbol{A} \,.\, \boldsymbol{a}_{\boldsymbol{\theta}}$$

$$A_{\theta} = (A_x\boldsymbol{a}_{\boldsymbol{x}} + A_y\boldsymbol{a}_{\boldsymbol{y}} + A_z\boldsymbol{a}_{\boldsymbol{z}}) \,.\, \boldsymbol{a}_{\boldsymbol{\theta}} \qquad (1.64)$$

Equation 1.64 can be simplified with the help of Table 1-7

$$A_{\theta} = \cos\theta\cos\varphi A_{\mathrm{x}} + \cos\theta\sin\varphi A_{\mathrm{y}} - \sin\theta\, A_{\mathrm{Z}} \qquad (1.65)$$

And $$A_{\emptyset} = -\sin\varphi\, A_{\mathrm{x}} + \cos\varphi\, A_{\mathrm{y}} + 0A_{\mathrm{Z}} \tag{1.66}$$

Equations 1.63, 1.65 and 1.66 can be written in the matrices format as

$$\begin{bmatrix} A_{\mathrm{r}} \\ A_{\theta} \\ A_{\varphi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} A_{\mathrm{x}} \\ A_{\mathrm{y}} \\ A_{\mathrm{z}} \end{bmatrix}$$

We will compute these three spherical components of Vector $\boldsymbol{A}$ and then the same vector in spherical coordinate system can be written as

$$\boldsymbol{A} = A_{\mathrm{r}}\boldsymbol{a}_{\mathbf{r}} + A_{\theta}\boldsymbol{a}_{\boldsymbol{\theta}} + A_{\emptyset}\boldsymbol{a}_{\emptyset} \tag{1.67}$$

1-14 Transformation of a Vector from SCS to RCS

We want to transform a vector from Spherical Coordinate System into Rectangular Coordinate System. Consider a vector in the Spherical Coordinate System

$$\boldsymbol{A} = A_{\mathrm{r}}\boldsymbol{a}_{\mathbf{r}} + A_{\theta}\boldsymbol{a}_{\boldsymbol{\theta}} + A_{\emptyset}\boldsymbol{a}_{\emptyset} \tag{1.68}$$

The scalar component A_{x} is given by

$$A_{\mathrm{x}} = \boldsymbol{A} \,.\, \boldsymbol{a}_{\mathbf{x}}$$

$$A_{\mathrm{x}} = (A_{\mathrm{r}}\boldsymbol{a}_{\mathbf{r}} + A_{\theta}\boldsymbol{a}_{\boldsymbol{\theta}} + A_{\emptyset}\boldsymbol{a}_{\emptyset}) \,.\, \boldsymbol{a}_{\mathbf{x}} \tag{1.69}$$

Equation 1.69 can be simplified with the help of Tables 1-5, 1-6 and 1-7

$$A_{\mathrm{x}} = \sin\theta\cos\varphi A_{\mathrm{r}} + \cos\theta\cos\varphi\, A_{\theta} - \sin\varphi\, A_{\varphi} \tag{1.70}$$

The scalar component A_{y} is given by

$$A_{\mathrm{y}} = \boldsymbol{A} \,.\, \boldsymbol{a}_{\mathbf{y}}$$

$$A_{\mathrm{y}} = (A_{\mathrm{r}}\boldsymbol{a}_{\mathbf{r}} + A_{\theta}\boldsymbol{a}_{\boldsymbol{\theta}} + A_{\emptyset}\boldsymbol{a}_{\emptyset}) \,.\, \boldsymbol{a}_{\mathbf{y}} \tag{1.71}$$

Equation 1.71 can be simplified with the help of Tables 1-5, 1-6 and 1-7

$$A_{\mathrm{y}} = \sin\theta\sin\varphi\, A_{\mathrm{r}} + \cos\theta\sin\varphi A_{\theta} + \cos\varphi\, A_{\emptyset} \tag{1.72}$$

The scalar component A_{z} is given by

$$A_z = \boldsymbol{A} \,.\, \boldsymbol{a_z}$$

$$A_z = (A_r \boldsymbol{a_r} + A_\theta \boldsymbol{a_\theta} + A_\emptyset \boldsymbol{a_\emptyset}) \,.\, \boldsymbol{a_z} \qquad (1.73)$$

Equation 1.73 can be simplified with the help of Tables 1-5, 1-6 and 1-7

$$A_z = \cos\theta\, A_r - \sin\theta\, A_\theta + 0A_\varphi \qquad (1.74)$$

Equations 1.70, 1.72 and 1.74 can be written in the matrices format as

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix}$$

We will compute these three rectangular components of Vector $\boldsymbol{A}$ and then the same vector in rectangular coordinate system can be written as

$$\boldsymbol{A} = A_x \boldsymbol{a_x} + A_y \boldsymbol{a_y} + A_z \boldsymbol{a_z} \qquad (1.75)$$

Example 1-6:

Transform $\boldsymbol{A} = 3\boldsymbol{a_x}$ into Spherical Coordinate System at $P(2, 90^0, 45^0)$

Solution:

We use the following equations for the required transformation.

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix} = \begin{bmatrix} \sin 90\cos 45 & \sin 90\sin 45 & \cos 90 \\ \cos 90\cos 45 & \cos 90\sin 45 & -\sin 90 \\ -\sin 45 & \cos 45 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix} = \begin{bmatrix} 0.707 & 0.707 & 0 \\ 0 & 0 & -1 \\ -7.07 & 7.07 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

Simplifying the equations given in the matrices format, we obtain

$$\boldsymbol{A} = 2.12\boldsymbol{a}_{\mathbf{r}} + 0\boldsymbol{a}_{\boldsymbol{\theta}} - 2.12\boldsymbol{a}_{\emptyset}$$

1-15 Differential Volume in Rectangular Coordinate System

Consider three differential vectors along x, y and z-axis as shown in Figure 1-31.

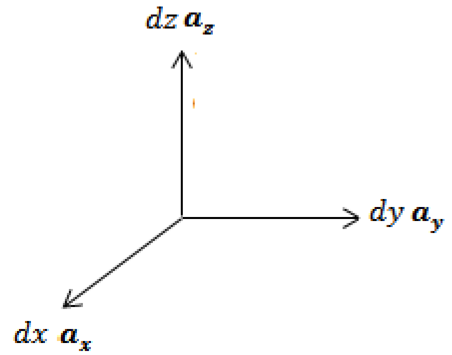


Figure 1-31: Differential Vectors

These differential length vectors are given by

$$\boldsymbol{dl} = dx\,\boldsymbol{a}_x$$

$$\boldsymbol{dl} = dy\,\boldsymbol{a}_y$$

$$\boldsymbol{dl} = dz\,\boldsymbol{a}_z$$

We construct a differential volume in rectangular coordinate system with the help of the magnitudes of the above three differential vectors as shown in Figure 1-32. The volume of the differential rectangular box is given by

$$dv = dx\,dy\,dz \tag{1.76}$$

The differential area in the direction of $\boldsymbol{a}_{\mathbf{x}}$ is given by

$$\boldsymbol{ds} = dy\,dz\,\boldsymbol{a}_{\mathbf{x}} \tag{1.77}$$

The differential area in the direction of $\boldsymbol{a}_{\mathbf{y}}$ is given by

$$\boldsymbol{ds} = dx\,dz\,\boldsymbol{a}_{\mathbf{y}} \tag{1.78}$$

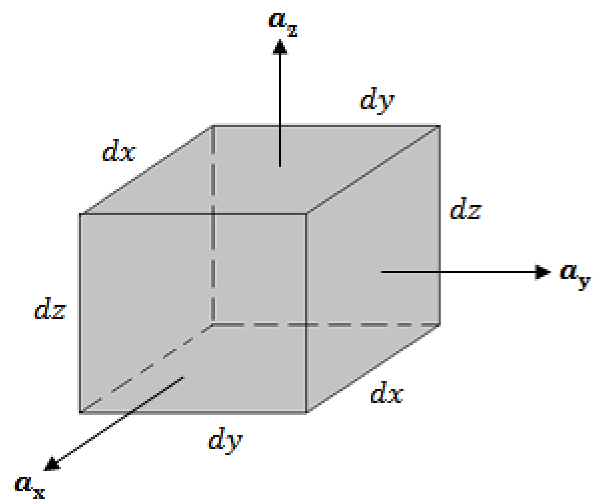


Figure 1- 32: Differential Volume

The differential area in the direction of $\boldsymbol{a_z}$ is given by

$$\boldsymbol{ds} = dx\ dy\ \boldsymbol{a_z} \tag{1.79}$$

A three dimensional differential length vector extending from point P to point Q in the rectangular box of Figure 1-33 is given by

$$\boldsymbol{dl} = dx\boldsymbol{a_x} + dy\boldsymbol{a_y} + dz\boldsymbol{a_z} \tag{1.80}$$

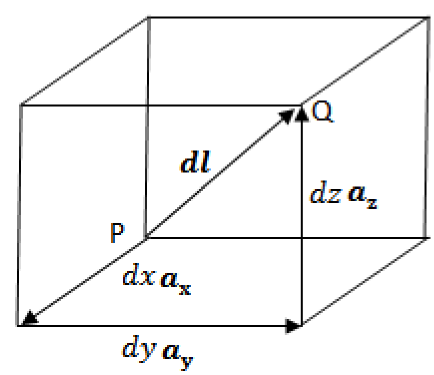


Figure 1-33: Differential Length Vector

1-16 Differential Volume in Cylindrical Coordinate System

Consider three differential vectors along $\boldsymbol{a_\rho}$, $\boldsymbol{a_\emptyset}$ and $\boldsymbol{a_z}$ as shown in Figure 1-34.

These differential length vectors are given by

$$\boldsymbol{dl} = d\rho \boldsymbol{a_\rho}$$

$$\boldsymbol{dl} = \rho d\varphi \boldsymbol{a_\varphi}$$

$$\boldsymbol{dl} = dz \boldsymbol{a_z}$$

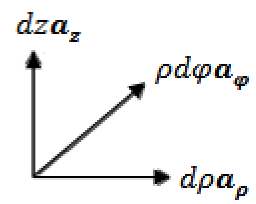


Figure 34: Differential Vectors

We construct a differential volume in cylindrical coordinate system with the help of the magnitudes of the above three differential vectors. The differential volume is shown in Figure 1-35.

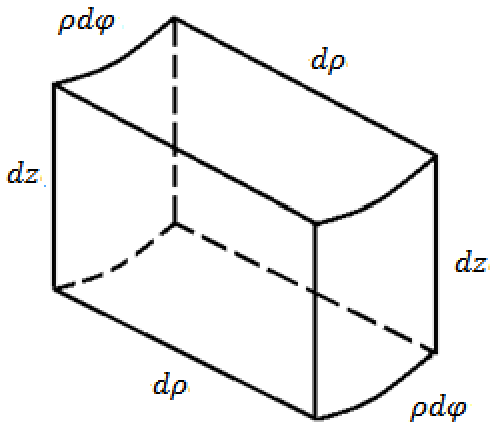


Figure 1-35: Differential Volume in Cylindrical Coordinate System

The volume of the differential rectangular box is given by

$$dv = d\rho \times \rho d\emptyset \times dz \tag{1.81}$$

Consider the differential surfaces in Figure 1-36. The differential area in the direction of

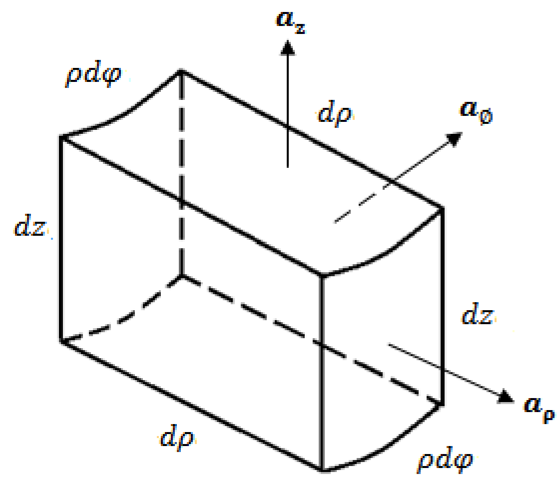


Figure 1-36: Differential Surfaces

$\boldsymbol{a}_\rho$ is given by

$$\boldsymbol{ds} = \rho d\emptyset \times dz\, \boldsymbol{a}_\rho \tag{1.82}$$

The differential area in the direction of $\boldsymbol{a}_\emptyset$ is given by

$$\boldsymbol{ds} = d\rho \times dz\, \boldsymbol{a}_\varphi \tag{1.83}$$

The differential area in the direction of $\boldsymbol{a}_z$ is given by

$$\boldsymbol{ds} = \rho d\rho \times d\emptyset \times\ \boldsymbol{a}_z \tag{1.84}$$

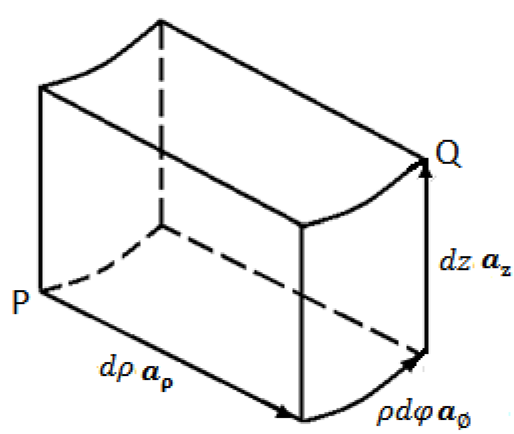


Figure 1-37: Differential Length Vector

A three dimensional differential length vector extending from P to Q in the differential cylindrical volume of Figure 1-37 is given by

$$\boldsymbol{dl} = d\rho\, \boldsymbol{a_\rho} + \rho d\emptyset\, \boldsymbol{a_\varphi} + dz\, \boldsymbol{a_z}$$

1-17 Differential Volume in Spherical Coordinate System

Consider three differential vectors along $\boldsymbol{a}_{\mathbf{r}}$, $\boldsymbol{a}_{\boldsymbol{\theta}}$ and $\boldsymbol{a}_{\emptyset}$ as shown in Figure 1-38. These differential length vectors are given by

$$\boldsymbol{dl} = dr\boldsymbol{a_r}$$

$$\boldsymbol{dl} = rd\theta\boldsymbol{a_\theta}$$

$$\boldsymbol{dl} = r\sin\theta d\varphi\, \boldsymbol{a_\varphi}$$

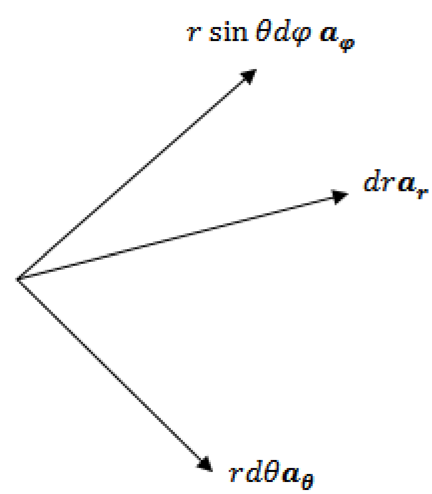


Figure 1-38: Differential Vectors

We construct a differential volume in spherical coordinate system with the help of the magnitudes of the above three differential vectors as shown in Figure 1-39.
The volume of the differential rectangular box is given by

$$dv = dr \times r\sin\theta d\varphi \times rd\theta$$

$$dv = r^2 \sin\theta\; d\theta\; dr\; d\varphi \qquad (1.85)$$

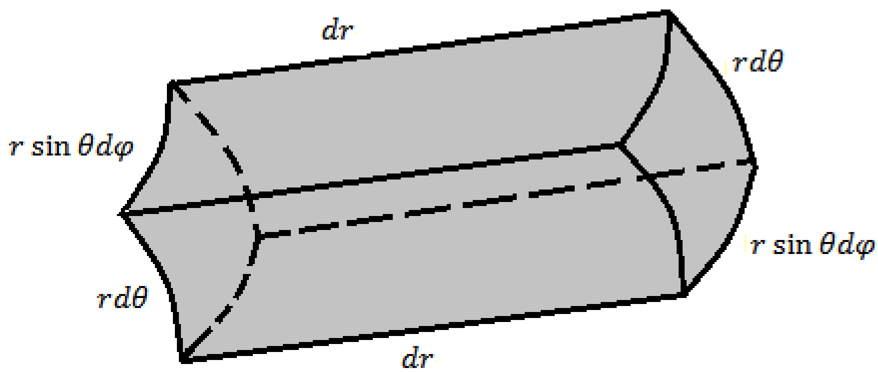


Figure 1-39: Differential Volume in Cylindrical Coordinate System

Consider the differential volume shown in Figure 1-40.

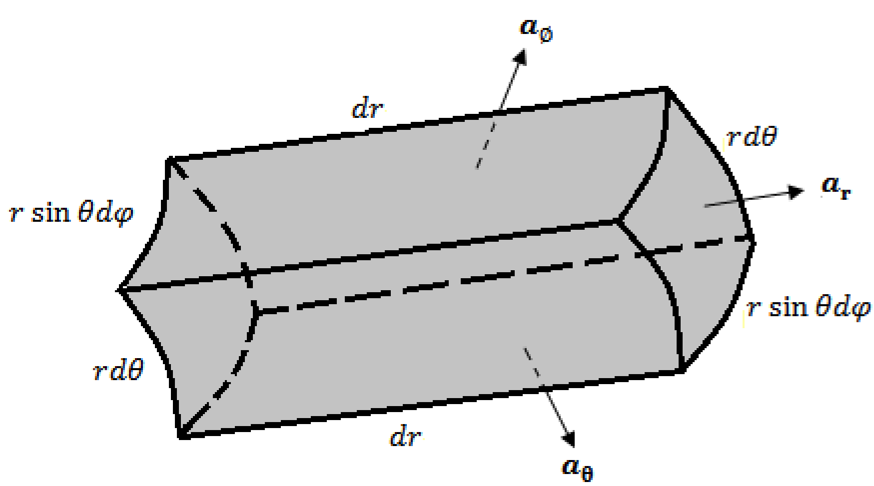


Figure 1-40: Differential Surfaces in Cylindrical Coordinate System

The differential area in the direction of $\boldsymbol{a}_\mathbf{r}$ is given by

$$\boldsymbol{ds} = r^2 \sin\theta \; d\theta \;\; d\varphi \;\; \boldsymbol{a}_\mathbf{r} \tag{1.86}$$

The differential area in the direction of $\boldsymbol{a}_\emptyset$ is given by

$$\boldsymbol{ds} = rdrd\theta \, \boldsymbol{a}_\boldsymbol{\varphi} \tag{1.87}$$

The differential area in the direction of $\boldsymbol{a}_\boldsymbol{\theta}$ is given by

$$\boldsymbol{ds} = r\sin\theta \; d\varphi \; dr \, \boldsymbol{a}_\boldsymbol{\theta} \tag{1.88}$$

A three dimensional differential length vector in the spherical coordinates system is given by

$$\boldsymbol{dl} = dr\ \boldsymbol{a_r} + r\sin\theta d\varphi\ \boldsymbol{a_\varphi} + rd\theta\boldsymbol{a_\theta}$$

1-18 Position Vector

Consider Figure 1-41, a vector extending from origin to point $P\ (x$, y, z) is known as position vector.

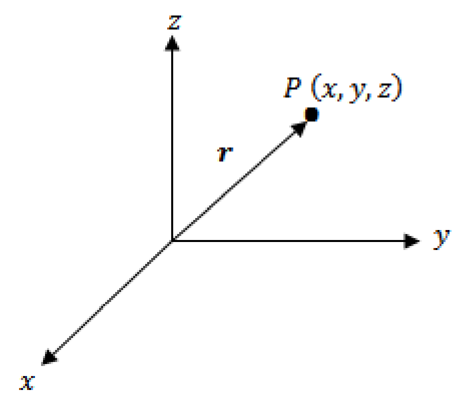


Figure 1-41: Position Vector $\boldsymbol{r}$

This three dimensional vector can be resolved into three components as shown in Figure 1-42.

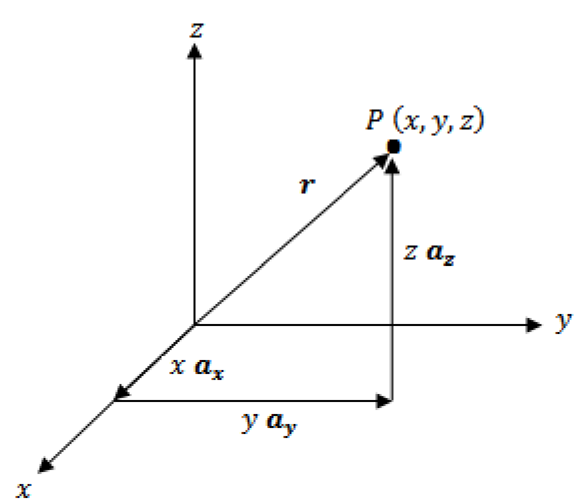


Figure 42: Components of Vector $\boldsymbol{r}$

$$\boldsymbol{r} = x\boldsymbol{a_x} + \ y\boldsymbol{a_y} + z\boldsymbol{a_z} \tag{1.89}$$

1-19 Distance Vector

Consider two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ as shown in Figure 1-43. The Vector extending from P to Q is known as distance vector that is represesnted by $\boldsymbol{R_{PQ}}$. Position vectors $\boldsymbol{r_P}$ and $\boldsymbol{r_Q}$ are given by

$$\boldsymbol{r_P} = x_1\boldsymbol{a_x} + \ y_1\boldsymbol{a_y} + z_1\,\boldsymbol{a_z}$$

$$\boldsymbol{r_Q} = x_2\boldsymbol{a_x} + \ y_2\boldsymbol{a_y} + z_2\,\boldsymbol{a_z}$$

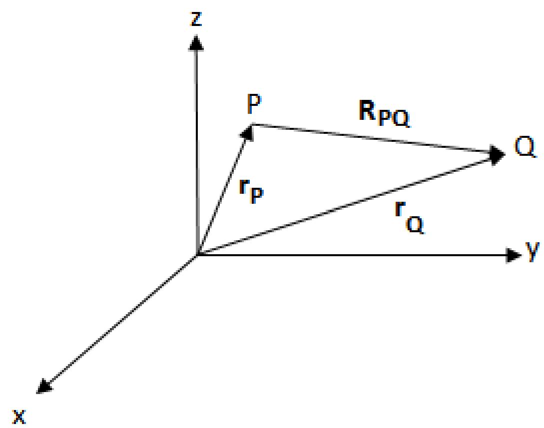


Figure 1-43: Distance Vector $\boldsymbol{R_{PQ}}$

The vector sum of $\boldsymbol{r_P}$ and $\boldsymbol{R_{PQ}}$ results in $\boldsymbol{r_Q}$.

$$\boldsymbol{r_P} + \boldsymbol{R_{PQ}} = \boldsymbol{r_Q}$$

$$\boldsymbol{R_{PQ}} = \boldsymbol{r_Q} - \boldsymbol{r_P}$$

$$\boldsymbol{R_{PQ}} = (x_2 - x_1)\boldsymbol{a_x} + \ (y_2 - y_1)\boldsymbol{a_y} + (z_2 - z_1)\boldsymbol{a_z} \tag{1.90}$$

The distance between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$R_{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{1.91}$$

Example 1-7:

Consider two points P and Q.

$$P(2,3,4) \qquad , \qquad Q(4,5,7)$$

Find (a) $\boldsymbol{r_P}$ (b) $\boldsymbol{r_Q}$ (c) $\boldsymbol{R_{PQ}}$ (d) $\boldsymbol{a_{PQ}}$

Solution:

(a)

$$\boldsymbol{r_P} = 2\boldsymbol{a_x} + 3\boldsymbol{a_y} + 4\boldsymbol{a_z}$$

(b)

$$\boldsymbol{r_Q} = 4\boldsymbol{a_x} + 5\boldsymbol{a_y} + 7\boldsymbol{a_z}$$

(c)

$$\boldsymbol{R_{PQ}} = (4-2)\boldsymbol{a_x} + (5-3)\boldsymbol{a_y} + (7-4)\boldsymbol{a_z}$$

$$\boldsymbol{R_{PQ}} = 2\boldsymbol{a_x} + 2\boldsymbol{a_y} + 3\boldsymbol{a_z}$$

(d)

$$R_{PQ} = \sqrt{4+4+9} = 4.12$$

$$\boldsymbol{a_{PQ}} = 0.485\boldsymbol{a_x} + 0.485\boldsymbol{a_y} + 0.728\boldsymbol{a_z}$$

Chapter 2

Force and Electric Field Intensity

2-1 Coulomb's Law

Consider two point charge particles Q_1 and Q_2 as shown in Figure 2-1. Position vector of Q_1 is $\boldsymbol{r_1}$ and position vector of Q_2 is $\boldsymbol{r_2}$. The distance vector extending from Q_1 to Q_2 is $\boldsymbol{R_{12}}$. The magnitude of this distance vector is R. Coulomb's law states that the electric force between the two charge particles is directly proportional to the product of Q_1 and Q_2 and is inversely proportional to square of the distance between them.

$$F \propto \frac{Q_1 Q_2}{R^2}$$

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon R^2}$$

There is a force of attraction between two unlike charge particles and a force of repulsion between two like charge particles. Force on Q_2 due to Q_1 is given by

$$\boldsymbol{F_2} = \frac{Q_1 Q_2}{4\pi\varepsilon R^2}\boldsymbol{a_R}$$

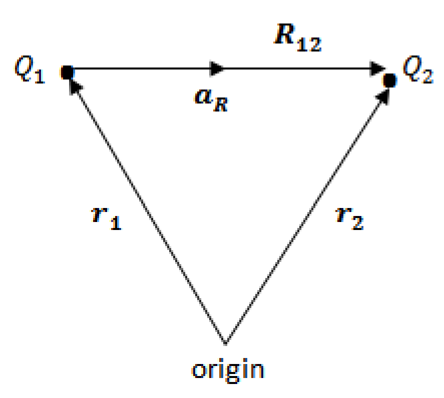


Figure 2-1: Two Point Charges

The force $\boldsymbol{F_2}$ can be written as

$$\boldsymbol{F_2} = \frac{Q_1 Q_2 \boldsymbol{R_{12}}}{4\pi\varepsilon\, R^3}$$

As $$\boldsymbol{R_{12}} = \boldsymbol{r_2} - \boldsymbol{r_1}$$

And $$R = |\boldsymbol{r_2} - \boldsymbol{r_1}|$$

Therefore

$$\boldsymbol{F_2} = \frac{Q_1 Q_2\, (\boldsymbol{r_2} - \boldsymbol{r_1})}{4\pi\varepsilon\, |\boldsymbol{r_2} - \boldsymbol{r_1}|^3} \qquad (2.1)$$

Force on Q_1 due to Q_2 is given by

$$\boldsymbol{F_1} = -\boldsymbol{F_2}$$

If we have n charge particles as shown in Figure 2-1a, then we apply Superposition Theorem to find the total force on the charge of Q coulomb.

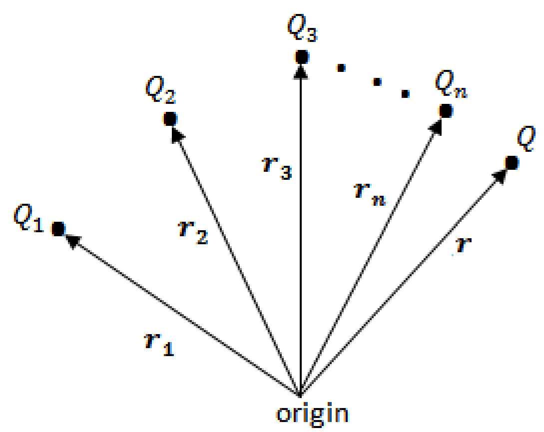


Figure 2-1a: n Charges

Assume that only charge Q_1 exists in the vicinity of the charge of Q coulomb and all other charges do not exist at all. Force on the charge of Q coulomb due to Q_1 is given by

$$\boldsymbol{F_1} = \frac{Q Q_1\, (\boldsymbol{r} - \boldsymbol{r_1})}{4\pi\varepsilon\, |\boldsymbol{r} - \boldsymbol{r_1}|^3}$$

Force on the charge of Q coulomb due to Q_2 is given by

$$\boldsymbol{F_2} = \frac{QQ_2\,(\boldsymbol{r}-\boldsymbol{r_2})}{4\pi\varepsilon\,|\boldsymbol{r}-\boldsymbol{r_2}|^3}$$

Finally force on the charge of Q coulomb due to Q_n is given by

$$\boldsymbol{F_n} = \frac{QQ_n\,(\boldsymbol{r}-\boldsymbol{r_n})}{4\pi\varepsilon\,|\boldsymbol{r}-\boldsymbol{r_n}|^3}$$

The vector sum of all these forces results in the total total force on Q.

$$\boldsymbol{F} = \boldsymbol{F_1} + \boldsymbol{F_2} + \boldsymbol{F_3} + \cdots + \boldsymbol{F_n}$$

$$\boldsymbol{F} = \frac{Q}{4\pi\varepsilon}\left[\frac{Q_1\,(\boldsymbol{r}-\boldsymbol{r_1})}{|\,\boldsymbol{r}-\boldsymbol{r_1}|^3} + \frac{Q_2\,(\boldsymbol{r}-\boldsymbol{r_2})}{|\,\boldsymbol{r}-\boldsymbol{r_2}|^3} + \frac{Q_3\,(\boldsymbol{r}-\boldsymbol{r_3})}{|\,\boldsymbol{r}-\boldsymbol{r_3}|^3} + \cdots \frac{Q_n\,(\boldsymbol{r}-\boldsymbol{r_n})}{|\,\boldsymbol{r}-\boldsymbol{r_n}|^3}\right] \qquad (A)$$

The last equation in concise form is given by

$$\boldsymbol{F} = \frac{Q}{4\pi\varepsilon}\sum_{i=1}^{n}\frac{Q_i\,(\boldsymbol{r}-\boldsymbol{r_i})}{|\,\boldsymbol{r}-\boldsymbol{r_i}|^3}$$

Example 2.1:

$Q_1 = \frac{1}{9}\mu C$ is located at the origin and $Q_2 = 100mC$ is located at $P(8,6,0)$ in free space. Find $\boldsymbol{F_2}$ and $\boldsymbol{F_1}$.

Solution:

$$\boldsymbol{F_2} = \frac{Q_1Q_2\,(\boldsymbol{r_2}-\boldsymbol{r_1})}{4\pi\varepsilon\,|\boldsymbol{r_2}-\boldsymbol{r_1}|^3}$$

$$\boldsymbol{r_2}-\boldsymbol{r_1} = 8\boldsymbol{a_x} + 6\boldsymbol{a_y} + 0\boldsymbol{a_z}$$

$$|\boldsymbol{r_2}-\boldsymbol{r_1}| = 10$$

$$\boldsymbol{F_2} = \frac{9\times 10^9 \times \frac{\mathbf{1}}{\mathbf{9}} \times 100 \times 10^{-9}\,(8\boldsymbol{a_x} + 6\boldsymbol{a_y})}{1000}$$

$$\boldsymbol{F_2} = 0.8\boldsymbol{a_x} + 0.6\boldsymbol{a_y}\ N$$

$$\boldsymbol{F_1} = -0.8\boldsymbol{a_x} - 0.6\boldsymbol{a_y}\ N$$

2-2 Electric Field Intensity due to a Point Charge

Consider two point charge particles Q_1 and Q as shown in Figure 2-2. Position vector of the source Q_1 is $\boldsymbol{r_1}$ and position vector of Q is $\boldsymbol{r}$. The distance vector extending from Q_1 to Q is $\boldsymbol{R}$. The magnitude of this distance vector is R.

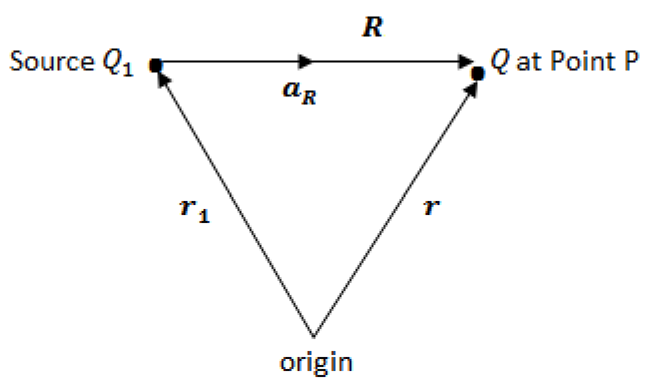


Figure 2-2: Two Point Charges

The electric force on charge of Q coulomb in accordance with Coulomb's law is given by.

$$\boldsymbol{F} = \frac{Q_1 Q}{4\pi\varepsilon R^2}\boldsymbol{a_R}$$

We want to find out the electric field intensity at point P that is caused by the source Q_1. The force on a charge of 1 coulomb at point P is known as electric field intensity. In other words force per unit charge is known as electric field intensity. It is a vector quantity and is always directed along the straight line joining the source and point P. Electric field intensity is given by

$$\boldsymbol{E} = \frac{F}{Q}$$

$$\boldsymbol{E} = \frac{Q_1}{4\pi\varepsilon R^2}\boldsymbol{a_R} \qquad (2.2)$$

or

$$\boldsymbol{E} = \frac{Q_1 \boldsymbol{R}}{4\pi\varepsilon R^3} \tag{2.3}$$

As $$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r_1}$$

And $$R = |\boldsymbol{r} - \boldsymbol{r_1}|$$

Therefore

$$\boldsymbol{E} = \frac{Q_1\,(\boldsymbol{r} - \boldsymbol{r_1})}{4\pi\varepsilon\,|\,\boldsymbol{r} - \boldsymbol{r_1}|^3} \tag{2.4}$$

This is the mathematical expression for electric field intensity at point P that is caused by the source Q_1.

2-3 Electric Field Intensity due to n Point Charges

Consider n charge particles as shown in Figure 2-3. Position vector of Q_1 is $\boldsymbol{r_1}$ and position vector of Q_2 is $\boldsymbol{r_2}$ and so on the position vector of Q_n is $\boldsymbol{r_n}$. We want to find out the electric field intensity at point P that is caused by all these n number of sources. Position vector of P is $\boldsymbol{r}$. In order to find out the electric field intensity at point P that is caused by all these n number of sources, we apply Superposition Theorem.

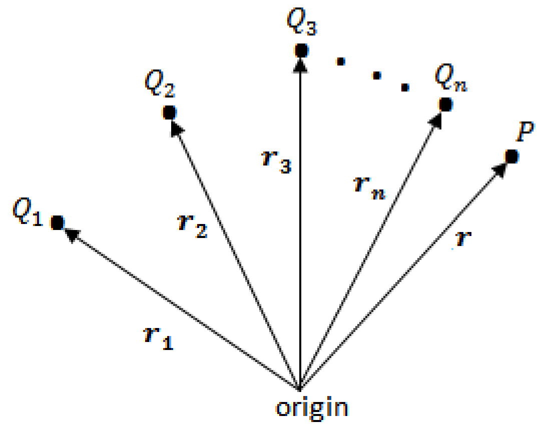


Figure 2-3: n Charge Particles

We assume that there is only one charge that is Q_1 in the vicinity of point P and all the remaining sources do not exist as shown in Figure 2-4. The distance vector extending from Q_1 to P is $\boldsymbol{r} - \boldsymbol{r_1}$.

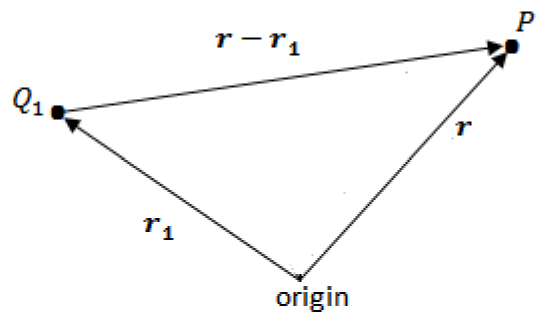


Figure 2-4: Intensity due to Q_1

Electric field intensity due to Q_1 is given by

$$\boldsymbol{E_1} = \frac{Q_1\,(\boldsymbol{r} - \boldsymbol{r_1})}{4\pi\varepsilon\,|\,\boldsymbol{r} - \boldsymbol{r_1}|^3} \tag{2.5}$$

Now, we assume that there is only one charge that is Q_2 in the vicinity of point P and all the remaining sources do not exist as shown in Figure 2-5. The distance vector extending from Q_2 to P is $\boldsymbol{r} - \boldsymbol{r_2}$.

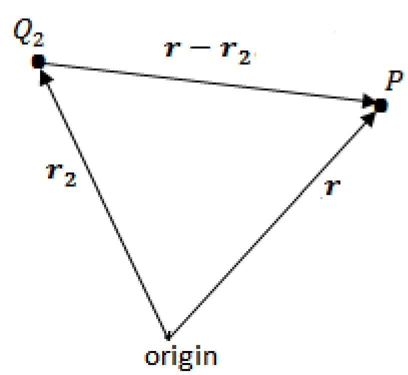


Figure 2-5: Intensity due to Q_2

Electric field intensity due to Q_2 is given by

$$\boldsymbol{E_2} = \frac{Q_2\,(\boldsymbol{r}-\boldsymbol{r_2})}{4\pi\varepsilon\,|\,\boldsymbol{r}-\boldsymbol{r_2}|^3} \tag{2.6}$$

Now, let us assume that there is only one charge that is Q_3 in the vicinity of point P and all the remaining sources do not exist as shown in Figure 2-6. The distance vector extending from Q_3 to P is $\boldsymbol{r}-\boldsymbol{r_3}$.

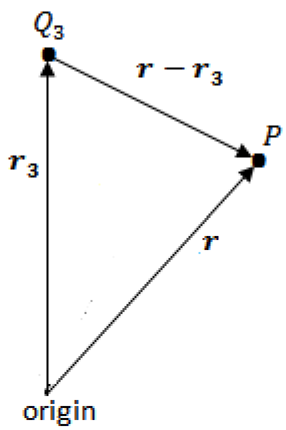


Figure 2-6: Intensity due to Q_3

Electric field intensity due to Q_3 is given by

$$\boldsymbol{E_3} = \frac{Q_3\,(\boldsymbol{r}-\boldsymbol{r_3})}{4\pi\varepsilon\,|\,\boldsymbol{r}-\boldsymbol{r_3}|^3} \tag{2.7}$$

Similarly electric field intensity at point P due to Q_n is given by

$$\boldsymbol{E_n} = \frac{Q_n\,(\boldsymbol{r}-\boldsymbol{r_n})}{4\pi\varepsilon\,|\,\boldsymbol{r}-\boldsymbol{r_n}|^3} \tag{2.8}$$

The vector sum of all these intensities results in the total electric field intensity at point P.

$$\boldsymbol{E} = \boldsymbol{E_1} + \boldsymbol{E_2} + \boldsymbol{E_3} + \cdots + \boldsymbol{E_n}$$

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon}\left[\frac{Q_1\,(\boldsymbol{r}-\boldsymbol{r_1})}{|\,\boldsymbol{r}-\boldsymbol{r_1}|^3} + \frac{Q_2\,(\boldsymbol{r}-\boldsymbol{r_2})}{|\,\boldsymbol{r}-\boldsymbol{r_2}|^3} + \frac{Q_3\,(\boldsymbol{r}-\boldsymbol{r_3})}{|\,\boldsymbol{r}-\boldsymbol{r_3}|^3} + \cdots \frac{Q_n\,(\boldsymbol{r}-\boldsymbol{r_n})}{|\,\boldsymbol{r}-\boldsymbol{r_n}|^3}\right] \tag{2.9}$$

The last equation in concise form is given by

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{n}\frac{Q_i\,(\boldsymbol{r}-\boldsymbol{r_i})}{|\,\boldsymbol{r}-\boldsymbol{r_i}|^3} \tag{2.10}$$

Example 2-2:

Find **E** in free space at $P(3,0,0)$ caused by

(i) $Q_1 = 9nC$ located at the origin.

(ii) $Q_2 = 125nC$ located at $(0,0,4)$.

(iii) Both Q_1 and Q_2

Solution:

(i)

$$\boldsymbol{E_1} = \frac{Q_1\,(\boldsymbol{r}-\boldsymbol{r_1})}{4\pi\varepsilon\,|\,\boldsymbol{r}-\boldsymbol{r_1}|^3}$$

$$\boldsymbol{r}-\boldsymbol{r_1} = 3\boldsymbol{a_x}$$

$$|\boldsymbol{r}-\boldsymbol{r_1}| = 3$$

$$\boldsymbol{E_1} = \frac{9\times10^9\ \times 9\times10^{-9}(3\boldsymbol{a_x})}{27}$$

$$\boldsymbol{E_1} = 9\boldsymbol{a_x}\ V/m$$

(ii)

$$\boldsymbol{E_2} = \frac{Q_2\,(\boldsymbol{r}-\boldsymbol{r_2})}{4\pi\varepsilon\,|\,\boldsymbol{r}-\boldsymbol{r_2}|^3}$$

$$\boldsymbol{r}-\boldsymbol{r_2} = 3\boldsymbol{a_x} - 4\,\boldsymbol{a_z}$$

$$|\boldsymbol{r}-\boldsymbol{r_2}| = 5$$

$$\boldsymbol{E_2} = \frac{9\times10^9\ \times 125\times10^{-9}(\,3\boldsymbol{a_x} - 4\,\boldsymbol{a_z})}{125}$$

$$E_2 = 27\boldsymbol{a}_x - 36\,\boldsymbol{a}_z \;V/m$$

(iii)

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2$$

$$\boldsymbol{E} = 36\boldsymbol{a}_x - 36\,\boldsymbol{a}_z \;V/m$$

2-4 Electric Field Intensity due to a Line Charge

Assume that charge is uniformly distributed along the length of a line as shown in Figure 2-7. Total charge on length L of the line is Q coulomb. Charge per unit length is known as line charge density which is represented by ρ_L. Unit of the line charge density is $C/_m$ and is given by

$$\rho_L = \frac{Q}{L}$$

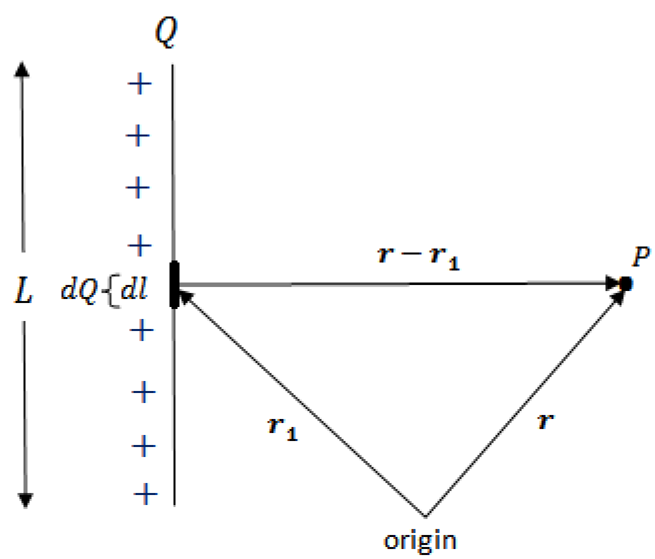


Figure 2-7: Intensity due to Line Charge

Consider a small portion of the line charge which is represented by $d\ell$. The magnitude of charge on this small portion of the line is dQ. So, the line charge density can be calculated with the help of following equation as well.

$$\rho_L = \frac{dQ}{d\ell}$$

Differential charge on the differential portion of the line is

$$dQ = \rho_L d\ell \tag{2.11}$$

The total charge of the siurce is given by

$$Q = \int \rho_L \; d\ell$$

In order to find out the electric field intensity at point P that is caused by the line charge, we consider small portion of the source first and determine the intensity at P that is caused by the small portion. Position vector of the small portion of the source is $\boldsymbol{r_1}$ and position vector of Point P is $\boldsymbol{r}$. The distance vector extending from dQ to P is $(\boldsymbol{r} - \boldsymbol{r_1})$. Differential electric field intensity at point P that is caused by the differential charge is given by

$$\boldsymbol{dE} = \frac{dQ\;(\boldsymbol{r} - \boldsymbol{r_1})}{4\pi\varepsilon\,|\,\boldsymbol{r} - \boldsymbol{r_1}|^3} \tag{2.12}$$

Putting the value of dQ in equation 2.12, we obtain

$$\boldsymbol{dE} = \frac{\rho_L d\ell\;(\boldsymbol{r} - \boldsymbol{r_1})}{4\pi\varepsilon\,|\,\boldsymbol{r} - \boldsymbol{r_1}|^3} \tag{2.13}$$

Total intensity that is caused by the entire source

$$\boldsymbol{E} = \int \frac{\rho_L d\ell\;(\boldsymbol{r} - \boldsymbol{r_1})}{4\pi\varepsilon\,|\,\boldsymbol{r} - \boldsymbol{r_1}|^3} \tag{2.14}$$

2-5 Electric Field Intensity due to Surface Charge

Assume that charge is uniformly distributed along surface of a sheet as shown in Figure 2-8. Total charge on surface S of the sheet is Q coulomb. Charge per unit area is known as surface charge density which is represented by ρ_S. Unit of the surface charge density is ${}^{C}/_{m^2}$ and is given by

$$\rho_S = \frac{Q}{S}$$

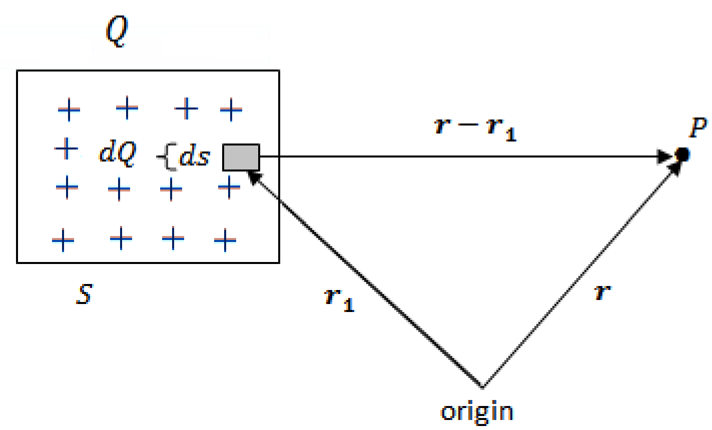


Figure 2-8: Intensity due to Surface Charge

Consider a small portion of the surface charge which is represented by ds. The magnitude of charge on this small portion of the source is dQ. So, the surface charge density can be calculated with the help of following equation as well.

$$\rho_S = \frac{dQ}{ds}$$

Differential charge on the differential portion of the surface is

$$dQ = \rho_S \, ds \tag{2.15}$$

The total charge of the siurce is given by

$$Q = \int \rho_s \, ds$$

In order to find out the electric field intensity at point P that is caused by the surface charge, we consider small portion of the source first and determine the intensity at P that is caused by the small portion. Position vector of the small portion of the source is $\boldsymbol{r_1}$ and position vector of Point P is $\boldsymbol{r}$. The distance vector extending from $d\,Q$ to P

is $(\boldsymbol{r}-\boldsymbol{r_1})$. Differential electric field intensity at point P that is caused by the differential charge is given by

$$\boldsymbol{dE} = \frac{dQ\,(\boldsymbol{r}-\boldsymbol{r_1})}{4\pi\varepsilon\,|\,\boldsymbol{r}-\boldsymbol{r_1}|^3} \qquad (2.16)$$

Putting the value of dQ in equation 2.16, we obtain

$$\boldsymbol{dE} = \frac{\rho_S\,ds\,(\boldsymbol{r}-\boldsymbol{r_1})}{4\pi\varepsilon\,|\,\boldsymbol{r}-\boldsymbol{r_1}|^3} \qquad (2.17)$$

Total intensity that is caused by the entire source

$$\boldsymbol{E} = \int \frac{\rho_S\,ds\,(\boldsymbol{r}-\boldsymbol{r_1})}{4\pi\varepsilon\,|\,\boldsymbol{r}-\boldsymbol{r_1}|^3} \qquad (2.18)$$

2-6 Electric Field Intensity due to Volume Charge

Assume that charge is uniformly distributed inside a volume V as shown in Figure 2-9. Total charge inside the given volume is Q coulomb. Charge per unit volume is known as volume charge density which is represented by ρ_v.

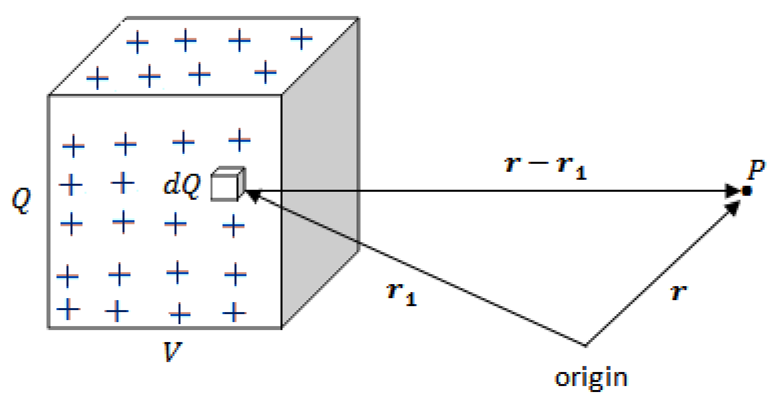


Figure 2-9: Intensity due to Volume Charge

Unit of the volume charge density is $^{C}/_{m^3}$ and is given by

$$\rho_v = \frac{Q}{V}$$

Consider a small portion of the volume charge which is represented by dv. The magnitude of charge in this small portion of the source is dQ. So, the volume charge density can be calculated with the help of following equation as well.

$$\rho_v = \frac{dQ}{dv}$$

Differential charge in the differential portion of the volume is

$$dQ = \rho_v \, dv \qquad (2.19)$$

The total charge of the siurce is given by

$$Q = \int \rho_v \; dv$$

In order to find out the electric field intensity at point P that is caused by the volume charge, we consider small portion of the source first and determine the intensity at P that is caused by the small portion. Position vector of the small portion of the source is $\boldsymbol{r_1}$ and position vector of Point P is $\boldsymbol{r}$**.** The distance vector extending from dQ to P is $(\boldsymbol{r} - \boldsymbol{r_1})$**.** Differential electric field intensity at point P that is caused by the differential charge is given by

$$\boldsymbol{dE} = \frac{dQ\ (\boldsymbol{r} - \boldsymbol{r_1})}{4\pi\varepsilon\,|\,\boldsymbol{r} - \boldsymbol{r_1}|^3} \qquad (2.20)$$

Putting the value of dQ in equation 2.20, we obtain

$$\boldsymbol{dE} = \frac{\rho_v\, dv\ (\boldsymbol{r} - \boldsymbol{r_1})}{4\pi\varepsilon\,|\,\boldsymbol{r} - \boldsymbol{r_1}|^3} \qquad (2.21)$$

Total intensity that is caused by the entire source

$$\boldsymbol{E} = \int \frac{\rho_v\, dv\ (\boldsymbol{r} - \boldsymbol{r_1})}{4\pi\varepsilon\,|\,\boldsymbol{r} - \boldsymbol{r_1}|^3} \qquad (2.22)$$

2-7 Electric Field Intensity due to Infinite Line Charge

Consider an infinite line charge extending from $-\infty$ to ∞ along $z-axis$ as shown in Figure 2-10. Line charge density of the source is ρ_L. Length of the small portion of the source is $d\ell$. Position vector of the small portion of the source is $\boldsymbol{r_1}$ and position vector of Point P is $\boldsymbol{r}$. The distance vector extending from $d\ell$ to P is $(\boldsymbol{r}-\boldsymbol{r_1})$. Electric field intensity at point P that is caused by aline charge is given by

$$\boldsymbol{E} = \int \frac{\rho_L d\ell\,(\boldsymbol{r}-\boldsymbol{r_1})}{4\pi\varepsilon\,|\,\boldsymbol{r}-\boldsymbol{r_1}|^3} \tag{2.23}$$

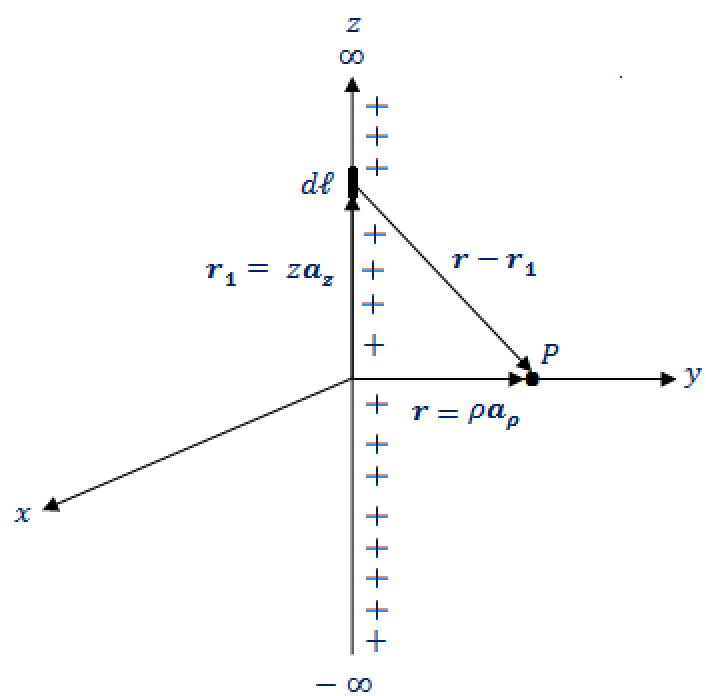


Figure 2-10: Intensity due to Infinite Line Charge

As $\qquad \boldsymbol{r}-\boldsymbol{r_1} = \rho\boldsymbol{a_\rho} - z\boldsymbol{a_z}$

And $\qquad |\,\boldsymbol{r}-\boldsymbol{r_1}| = \sqrt{\rho^2 + z^2} \quad , \; d\ell = dz$

Therefore electric field intensity at point P that is caused by the infinite line charge is given by

$$\boldsymbol{E} = \int_{-\infty}^{\infty} \frac{\rho_L dz\,(\rho \boldsymbol{a_\rho} - z\boldsymbol{a_z})}{4\pi\varepsilon\,(\rho^2 + z^2)^{3/2}} \tag{2.24}$$

Consider the right angle triangle as shown in Figure 2-11.

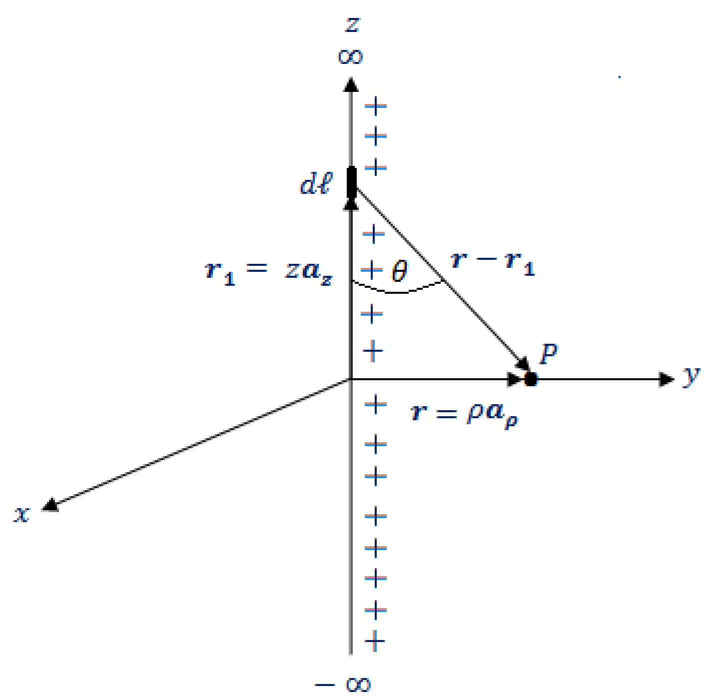


Figure 2-11: Infinite Line Charge

$$\frac{z}{\rho} = \cot\theta$$

$$z = \rho\cot\theta$$

$$dz = -\rho\,\mathrm{cosec}^2\,\theta\; d\theta$$

When $z = -\infty$, then $\theta = \pi$ and when $z = \infty$, then $\theta = 0$, putting all these values in equation 2.24, we obtain

$$\boldsymbol{E} = \int_0^\pi \frac{\rho_L\, \rho\ \operatorname{cosec}^2\theta\, d\theta (\rho \boldsymbol{a_\rho} - \rho \cot\theta\, \boldsymbol{a_z})}{4\pi\varepsilon\, (\rho^2 + \ \rho^2 \cot^2\theta)^{3/2}} \tag{2.25}$$

$$\boldsymbol{E} = \int_0^\pi \frac{\rho_L\, \rho^2 \operatorname{cosec}^2\theta\, d\theta\, (\boldsymbol{a_\rho} - \cot\theta\, \boldsymbol{a_z})}{4\pi\varepsilon\, \rho^3 (1 + \ \cot^2\theta\ )^{3/2}} \tag{2.26}$$

$$\boldsymbol{E} = \frac{\rho_L}{4\pi\varepsilon\rho} \int_0^\pi \frac{\operatorname{cosec}^2\theta\ d\theta\, (\boldsymbol{a_\rho} - \cot\theta\, \boldsymbol{a_z})}{(\operatorname{cosec}^2\ )^{3/2}} \tag{2.27}$$

$$\boldsymbol{E} = \frac{\rho_L}{4\pi\varepsilon\rho} \int_0^\pi \frac{d\theta\ (\boldsymbol{a_\rho} - \cot\theta\, \boldsymbol{a_z})}{cosec\ \theta} \tag{2.28}$$

$$\boldsymbol{E} = \frac{\rho_L}{4\pi\varepsilon\rho} \int_0^\pi \sin\theta\, d\theta\, (\boldsymbol{a_\rho} - \cot\theta\, \boldsymbol{a_z})$$

$$\boldsymbol{E} = \frac{\rho_L}{4\pi\varepsilon\rho} \left[\int_0^\pi \sin\theta\, d\theta\, \boldsymbol{a_\rho} - \int_0^\pi \cot\theta \sin\theta\, d\theta\, \boldsymbol{a_z} \right] \tag{2.29}$$

$$\boldsymbol{E} = \frac{\rho_L}{4\pi\varepsilon\rho} \left[\int_0^\pi \sin\theta\, d\theta\, \boldsymbol{a_\rho} - \int_0^\pi \cos\theta\, d\theta\, \boldsymbol{a_z} \right] \tag{2.30}$$

As $$\int_0^\pi \cos\theta\, d\theta\, \boldsymbol{a_z} = 0$$

Therefore $$\boldsymbol{E} = \frac{\rho_L}{4\pi\varepsilon\rho} \left[\int_0^\pi \sin\theta\, d\theta\, \boldsymbol{a_\rho} \right]$$

$$\boldsymbol{E} = \frac{\rho_L}{4\pi\varepsilon\rho} \left| -\cos\theta \right|_0^\pi \boldsymbol{a_\rho}$$

$$\boldsymbol{E} = \frac{\rho_L}{2\pi\varepsilon\rho}\boldsymbol{a_\rho} \qquad (2.31)$$

If we enclose the source in a cylinder as shown in Figure 2-12, electric field intensity on the surface of the cylinder will be constant. Similarly if we consider a line parallel to the source then electric field intensity along this lline will be constant.

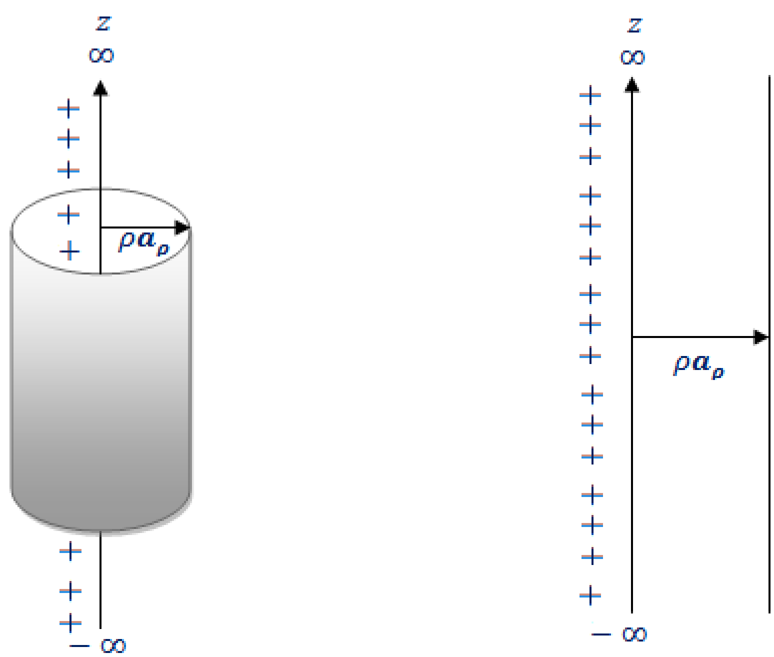


Figure 2-12: Intensity on the Surface of a Cylinder and along a line parallel to Source

If we replace ρ by R and $\boldsymbol{a_\rho}$ by $\boldsymbol{a_R}$ in equation 2.31 then the same equation can be written as

$$\boldsymbol{E} = \frac{\rho_L}{2\pi\varepsilon R}\boldsymbol{a_R} \qquad (2.32)$$

Example 2-3:

A line charge is extending from $z = -3$ to $z = 3$. If the line charge density of the source is $4nC/m$, then find $\boldsymbol{E}$ at $P(0, 4, 0)$.

Solution:

Consider the line charge in Figure 12a. Electric field intensity is given by

$$\boldsymbol{E} = \int \frac{\rho_L d\ell\,(\boldsymbol{r} - \boldsymbol{r_1})}{4\pi\varepsilon\,|\,\boldsymbol{r} - \boldsymbol{r_1}|^3} \qquad (A)$$

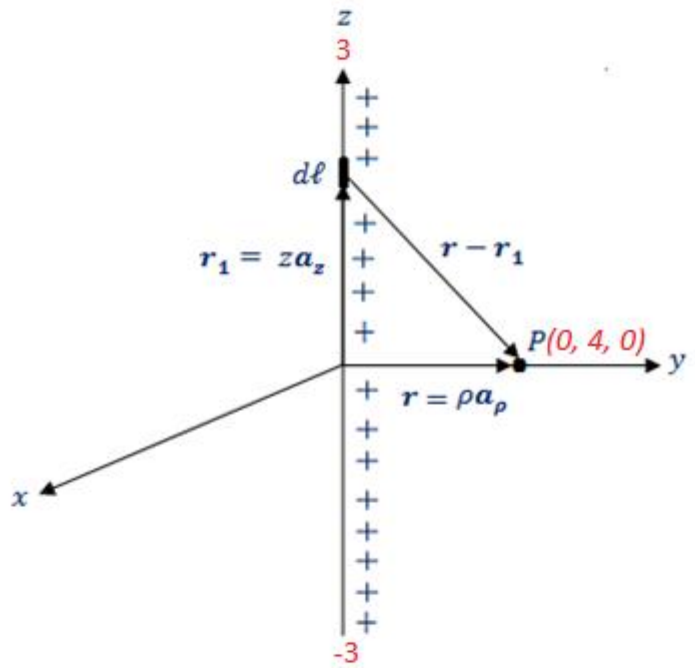


Figure 12a: Finite Line Charge

As $\qquad \boldsymbol{r} - \boldsymbol{r_1} = \rho\boldsymbol{a_\rho} - z\boldsymbol{a_z}$

And $\qquad |\,\boldsymbol{r} - \boldsymbol{r_1}| = \sqrt{\rho^2 + z^2} \quad , \; d\ell = dz$

Therefore electric field intensity at point P that is caused by the infinite line charge is given by

$$\boldsymbol{E} = \int_{-3}^{3} \frac{\rho_L dz\,(\rho\boldsymbol{a_\rho} - z\boldsymbol{a_z})}{4\pi\varepsilon\,(\rho^2 + z^2)^{3/2}} \qquad (B)$$

Let

$$\frac{z}{\rho} = \tan\theta$$

$$z = \rho \tan\theta$$

$$dz = \rho \sec^2\theta \, d\theta$$

When $z = -3$, then $\theta = -36.8$ and when $z = 3$, then $\theta = 36.8$, putting all these values in equation B, we obtain

$$\boldsymbol{E} = \int_{-36.8}^{36.8} \frac{\rho_L \, \rho \, \sec^2\theta \, d\theta (\rho \boldsymbol{a_\rho} - \rho \, tan\,\theta \, \boldsymbol{a_z})}{4\pi\varepsilon \, (\rho^2 + \rho^2 \, tan^2\,\theta)^{3/2}}$$

$$\boldsymbol{E} = \int_{-36.8}^{36.8} \frac{\rho_L \, \rho^2 \sec^2\theta \, d\theta \, (\boldsymbol{a_\rho} - \tan\theta \, \boldsymbol{a_z})}{4\pi\varepsilon \, \rho^3 (1 + \tan^2\theta \,)^{3/2}}$$

$$\boldsymbol{E} = \frac{\rho_L}{4\pi\varepsilon\rho} \int_{-36.8}^{36.8} \frac{\sec^2\theta \, d\theta \, (\boldsymbol{a_\rho} - \tan\theta \, \boldsymbol{a_z})}{(\sec^2\,)^{3/2}}$$

$$\boldsymbol{E} = \frac{\rho_L}{4\pi\varepsilon\rho} \int_{-36.8}^{36.8} \frac{d\theta \, (\boldsymbol{a_\rho} - \tan\theta \, \boldsymbol{a_z})}{sec\,\theta}$$

$$\boldsymbol{E} = \frac{\rho_L}{4\pi\varepsilon\rho} \int_{-36.8}^{36.8} \cos\theta \, d\theta \, (\boldsymbol{a_\rho} - \tan\theta \, \boldsymbol{a_z})$$

$$\boldsymbol{E} = \frac{\rho_L}{4\pi\varepsilon\rho} \left[\int_{-36.8}^{36.8} \cos\theta \, d\theta \, \boldsymbol{a_\rho} - \int_{-36.8}^{36.8} \tan\theta \cos\theta \, d\theta \, \boldsymbol{a_z} \right]$$

$$\boldsymbol{E} = \frac{\rho_L}{4\pi\varepsilon\rho} \left[\int_{-36.8}^{36.8} \cos\theta \, d\theta \, \boldsymbol{a_\rho} - \int_{-36.8}^{36.8} \sin\theta \, d\theta \, \boldsymbol{a_z} \right]$$

As $$\int_{-36.8}^{36.8} \sin\theta \, d\theta \, \boldsymbol{a_z} = 0$$

Therefore $$\boldsymbol{E} = \frac{\rho_L}{4\pi\varepsilon\rho}\left[\int_{-36.8}^{36.8} \cos\theta \, d\theta \, \boldsymbol{a_\rho}\right]$$

$$\boldsymbol{E} = \frac{\rho_L}{4\pi\varepsilon\rho}|\sin\theta|_{-36.8}^{36.8} \, \boldsymbol{a_\rho}$$

$$\boldsymbol{E} = \frac{9\times10^9 \times 4 \times 10^{-9}}{4}(0.6+0.6)\boldsymbol{a_\rho}$$

$$\boldsymbol{E} = 10.8 \, \boldsymbol{a_y} \quad V/m$$

2-8 Electric Field Intensity due to Infinite Sheet of Charge

Consider an infinite sheet of charge which is located in $x = 0$ plane as shown in Figure 2-13. The sheet is divided into a very large number of narrow strips and width of each strip is represented by dy. Width of a narrow strip is so small that it almost behaves like an infinite line charge. We want to find out electric field intensity at point P. consider the narrow strip on the right hand side that acts like an infinite line charge. The surface charge density of the sheet is given by

$$\rho_s = \frac{dQ}{dydz}$$

Line charge density of the infinite line charge is given by

$$\rho_L = \frac{dQ}{dz} = \rho_s dy \tag{2.33}$$

Distance vector extending from the infinite line charge to the point P is

$$\boldsymbol{R_1} = x\boldsymbol{a_x} - y\boldsymbol{a_y} \tag{2.34}$$

$$R_1 = \sqrt{x^2+y^2} \tag{2.35}$$

$$a_R = \frac{x\boldsymbol{a}_x - y\boldsymbol{a}_y}{\sqrt{x^2 + y^2}} \qquad (2.36)$$

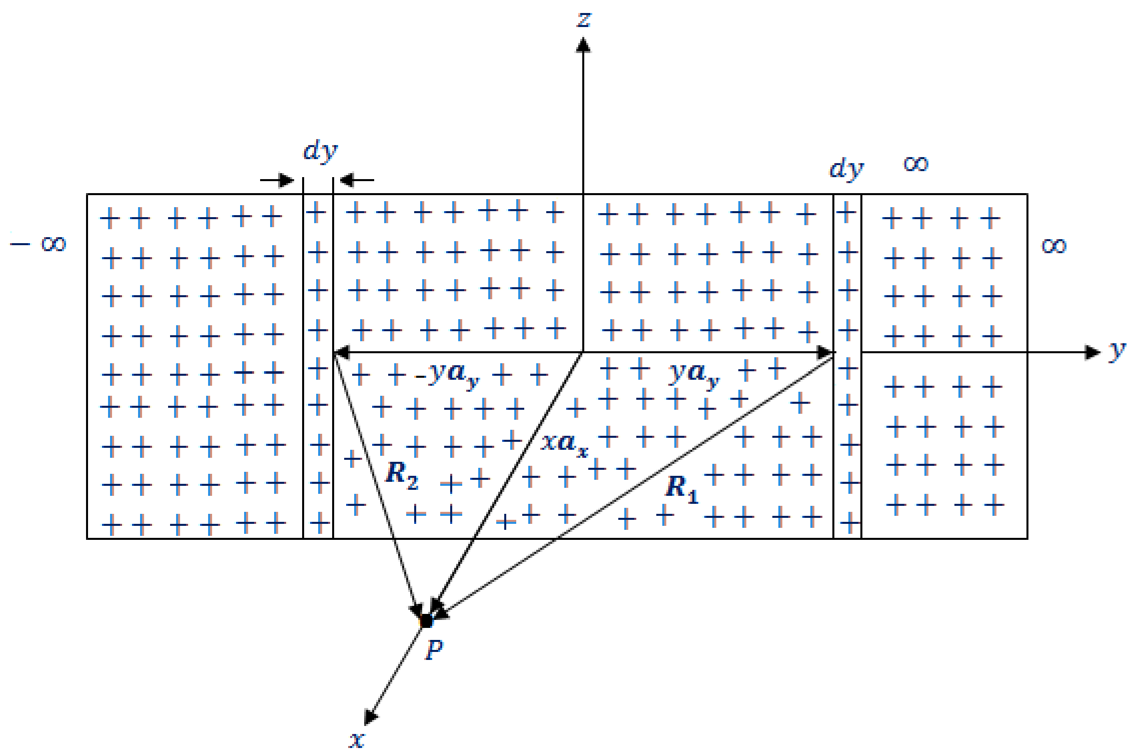


Figure 13: Intensity due to Infinite Sheet of Charge

Electric field intensity due to this infinite line charge is given by

$$\boldsymbol{E} = \frac{\rho_L}{2\pi\varepsilon R}\boldsymbol{a}_R \qquad (2.37)$$

Putting the values in equation 37, we obtain differential intensity due to the narrow Strip on right hand side.

$$\boldsymbol{dE} = \frac{\rho_s\, dy\, (x\boldsymbol{a}_x - y\boldsymbol{a}_y)}{2\pi\varepsilon\, (x^2 + y^2)} \qquad (2.38)$$

Due to symmetry, consider the narrow strip on the left hand side that acts like an infinite line charge as well. Distance vector extending from the left hand side infinite line charge to the point P is

$$\boldsymbol{R_2} = x\boldsymbol{a_x} + y\boldsymbol{a_y} \tag{2.39}$$

$$R_2 = \sqrt{x^2 + y^2} \tag{2.40}$$

$$\boldsymbol{a_R} = \frac{x\boldsymbol{a_x} + y\boldsymbol{a_y}}{\sqrt{x^2 + y^2}} \tag{2.41}$$

Electric field intensity due to this infinite line charge is calculated by putting values in equation 2.37.

$$\boldsymbol{dE} = \frac{\rho_S\, dy\, (x\boldsymbol{a_x} + y\boldsymbol{a_y})}{2\pi\varepsilon\, (x^2 + y^2)} \tag{2.42}$$

We add equation 2.38 and 2.42

$$\boldsymbol{dE} + \boldsymbol{dE} = \frac{\rho_S\, dy\, (x\boldsymbol{a_x} - y\boldsymbol{a_y})}{2\pi\varepsilon\, (x^2 + y^2)} + \frac{\rho_S\, dy\, (x\boldsymbol{a_x} - y\boldsymbol{a_y})}{2\pi\varepsilon\, (x^2 + y^2)}$$

Due to symmetry y components of the electric field intensity cancel the effect of each other, so we have

$$\boldsymbol{dE} = \frac{\rho_S\, dy\, x\boldsymbol{a_x}}{2\pi\varepsilon\, (x^2 + y^2)} \tag{2.43}$$

$$\boldsymbol{E} = \int_{-\infty}^{\infty} \frac{\rho_S\, dy\, x\boldsymbol{a_x}}{2\pi\varepsilon\, (x^2 + y^2)} \tag{2.44}$$

Consider the right angle triangle as shown in Figure 2-13.

$$\frac{y}{x} = \cot\theta$$

$$y = x\cot\theta$$

$$dy = -\, x\,\mathrm{cosec}^2\,\theta\; d\theta$$

When $y = -\infty$, then $\theta = \pi$ and when $y = \infty$, then $\theta = 0$, putting all these values in equation 2.44, we obtain

$$\boldsymbol{E} = \int_{\boldsymbol{0}}^{\boldsymbol{\pi}} \frac{\rho_s x^2 \operatorname{cosec}^2 \theta \, d\theta \, \boldsymbol{a_x}}{2\pi\varepsilon \, (x^2 + x^2 \cot^2 \theta)} \tag{2.45}$$

$$\boldsymbol{E} = \int_{\boldsymbol{0}}^{\boldsymbol{\pi}} \frac{\rho_s x^2 \operatorname{cosec}^2 \theta \, d\theta \, \boldsymbol{a_x}}{2\pi\varepsilon \, (x^2 \operatorname{cosec}^2 \theta)}$$

$$\boldsymbol{E} = \frac{\rho_s}{2\pi\varepsilon} \int_{\boldsymbol{0}}^{\boldsymbol{\pi}} d\theta \;\; \boldsymbol{a_x} \tag{2.46}$$

$$\boldsymbol{E} = \frac{\rho_s}{2\varepsilon} \boldsymbol{a_x} \tag{2.47}$$

Where $\boldsymbol{a_x}$ is a unit vector normal to the sheet. Let us represent this unit vector normal to the sheet by $\boldsymbol{a_n}$. Equation no 2.47 in generic form can be written as

$$\boldsymbol{E} = \frac{\rho_s}{2\varepsilon} \boldsymbol{a_n} \tag{2.48}$$

Consider two infinite sheets as shown in Figure 2-14. We apply Super position Theorem to find the total intensity at P.

Electric Field intensity due to positive sheet is given by

$$\boldsymbol{E}_+ = \frac{\rho_s}{2\varepsilon} \boldsymbol{a_n}$$

Electric Field intensity due to negative sheet is given by

$$\boldsymbol{E}_- = -\frac{\rho_s}{2\varepsilon} \boldsymbol{a_n}$$

Total Electric Field intensity at P is given by

$$\boldsymbol{E} = \boldsymbol{E}_+ + \boldsymbol{E}_- = \boldsymbol{0}$$

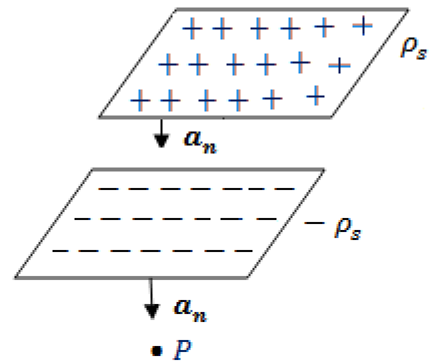


Figure 2-14: Intensity due to two Infinite Sheets of Charge

Consider second case of two infinite sheets as shown in Figure 2-15. We apply Super position Theorem to find the total intensity at P.

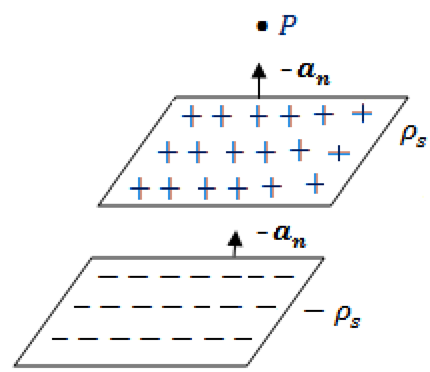


Figure 2-15: Intensity due to two Infinite Sheets of Charge

Electric Field intensity due to positive sheet is given by

$$\boldsymbol{E}_{+} = -\frac{\rho_s}{2\varepsilon}\boldsymbol{a_n}$$

Electric Field intensity due to negative sheet is given by

$$\boldsymbol{E}_{-} = \frac{\rho_s}{2\varepsilon}\boldsymbol{a_n}$$

Total Electric Field intensity at P is given by

$$\boldsymbol{E} = \boldsymbol{E}_+ + \boldsymbol{E}_- = \boldsymbol{0}$$

Consider third case of two infinite sheets as shown in Figure 2-16. We apply Super position Theorem to find the total intensity at P.

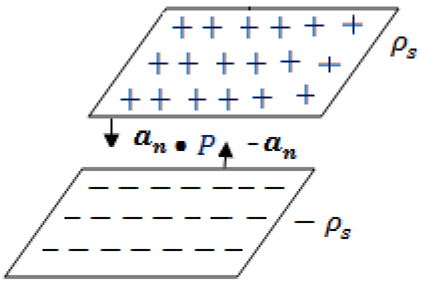


Figure 2-16: Intensity due to two Infinite Sheets of Charge

Electric Field intensity due to positive sheet is given by

$$\boldsymbol{E}_+ = \frac{\rho_s}{2\varepsilon}\boldsymbol{a_n}$$

Electric Field intensity due to negative sheet is given by

$$\boldsymbol{E}_- = \frac{\rho_s}{2\varepsilon}\boldsymbol{a_n}$$

Total Electric Field intensity at P is given by

$$\boldsymbol{E} = \boldsymbol{E}_+ + \boldsymbol{E}_- = \frac{\rho_s}{\varepsilon}\boldsymbol{a_n}$$

Example 2-4:

A surface charge is located at $x = 0$. If the surface charge density of the source is $17.75 \times 10^{-15}\ C/m^2$, then find $\boldsymbol{E}$ at $P(9, 5, 7)$ in free space.

Solution:

$$\boldsymbol{E} = \frac{\rho_s}{2\varepsilon}\boldsymbol{a_n}$$

$$\boldsymbol{E} = \frac{17.7 \times 10^{-15}}{17.7 \times 10^{-12}} \boldsymbol{a_x}$$

$$\boldsymbol{E} = \boldsymbol{a_x} \ mV/m$$

Example 2-5:

A surface charge of $17.75 \times 10^{-12} \frac{C}{m^2}$ is located at $x = -2$, surface charge density of another source is $17.7 \times 10^{-12} \ C/m^2$ that is located at $x = 2$. Find $\boldsymbol{E}$ at $P(0,0,0)$ and $\boldsymbol{E}$ at $P(4,0,0)$ in free space.

Solution

a.

$$\boldsymbol{E} = \frac{\rho_s}{2\varepsilon} \boldsymbol{a_x} - \frac{\rho_s}{2\varepsilon} \boldsymbol{a_x} = 0$$

b.

$$\boldsymbol{E} = \frac{\rho_s}{2\varepsilon} \boldsymbol{a_x} + \frac{\rho_s}{2\varepsilon} \boldsymbol{a_x} = 2\boldsymbol{a_x} \ V/m$$

Chapter 3

Electric Flux & Electric Flux Density

3-1 Electric Flux

Consider a point charge of Q coulomb that can not move from one place to another place as shown in Figure 3-1. There will be electric field in the vicinity of this charge. If we place a movable unit positive in the electric field of the fixed charge it will move along a straight line due the force of repulsion. The path or line followed by a unit positive charge in an electric field is known as electric flux. It is a scalar quantity and is represented by ψ. The unit of electric flux is coulomb.

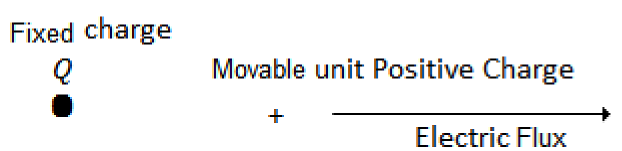


Figure 3-1: Electric Flux

We may change the place of the movable unit positive charge around the fixed one and can trace many more lines. The electric force, electric field intensity and electric flux density are in the direction of arrow, in other words the direction of the arrow gives the direction of the electric force, electric field intensity and electric flux density.

3-2 Electric Flux Density

Consider lines of electric force ψ passing through a surface S as shown in Figure 3-2. All the lines are normal to the surface. The electric flux per unit area defines electric flux density and it is represented by $\boldsymbol{D}$. It is a vector quantity and its unit is C/m^2.

Mathematically

$$D = \frac{\psi}{S}$$

So

$$\psi = DS$$

as

$$\boldsymbol{D} = D\boldsymbol{a_n}$$

$$\boldsymbol{S} = S\boldsymbol{a_n}$$

so

$$\psi = \boldsymbol{D}.\boldsymbol{S}$$

There is another way to compute electric flux density. We consider differential electric flux $d\psi$ passing through a small portion of the given surface that is ds. According to the definition, the electric flux per unit area can be calculated as

$$D = \frac{d\psi}{ds}$$

Hence the differential electric flux passing through the differential area can be computed as under

$$d\psi = Dds$$

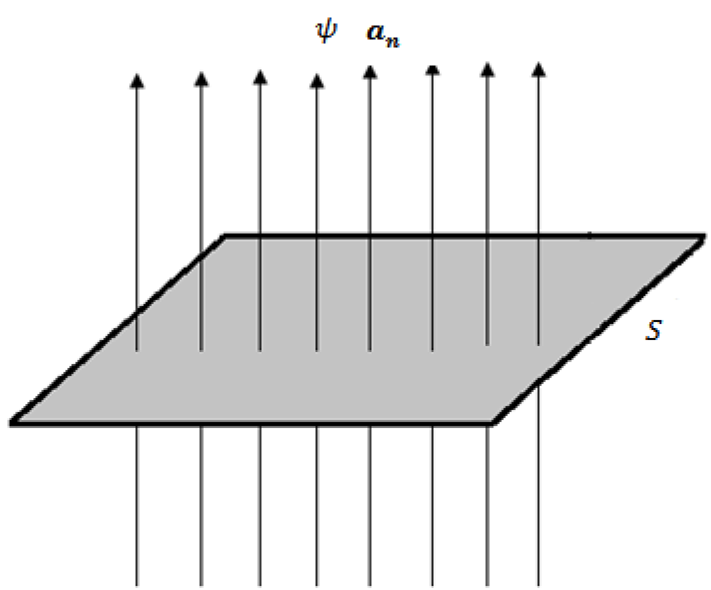


Figure 3-2: Electric Flux Density

As

$$\boldsymbol{ds} = ds\boldsymbol{a_n}$$

$$d\psi = \boldsymbol{D}.\boldsymbol{ds}$$

$$\psi = \int \boldsymbol{D}.\boldsymbol{ds}$$

The flux passing through the closed surface is given by

$$\psi = \oint \boldsymbol{D}.\boldsymbol{ds}$$

3-3 Electric Flux Density due to a Point Charge

Consider a point charge particle Q as shown in Figure 3-3. Position vector of the source Q is $\boldsymbol{r_1}$ and position vector of P is $\boldsymbol{r}$. The distance vector extending from Q to P is $\boldsymbol{R}$. The magnitude of this distance vector is R.

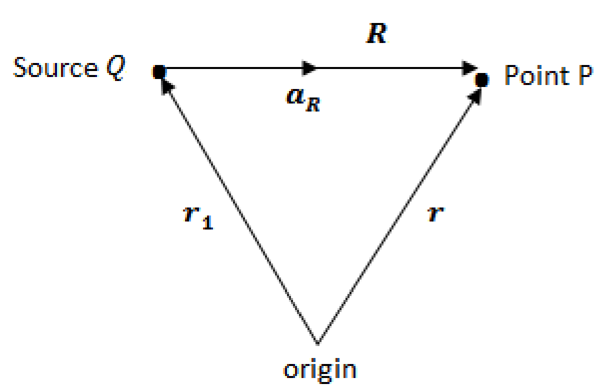


Figure 3-3: Point Charge

We want to find out the electric flux density at point P that is caused by the source Q. The force on a charge of 1 coulomb at point P is known as electric field intensity. In other words force per unit charge is known as electric field intensity.

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon R^2}\boldsymbol{a_R} \qquad (3.1)$$

$$\text{as} \qquad \boldsymbol{D} = \varepsilon \boldsymbol{E}$$

Electric flux density at point P is given by

$$\boldsymbol{D} = \frac{Q}{4\pi R^2} \boldsymbol{a_R} \tag{3.2}$$

It is a vector quantity and is always directed along the straight line joining the source and point P.

or

$$\boldsymbol{D} = \frac{Q\boldsymbol{R}}{4\pi R^3} \tag{3.3}$$

As $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r_1}$

And $R = |\boldsymbol{r} - \boldsymbol{r_1}|$

Therefore

$$\boldsymbol{D} = \frac{Q\,(\boldsymbol{r} - \boldsymbol{r_1})}{4\pi\,|\boldsymbol{r} - \boldsymbol{r_1}|^3} \tag{3.4}$$

This is the mathematical expression for electric fluxdensity at point P that is caused by the source Q.

3-4 Electric Flux Intensity due to n Point Charges

Consider n charge particles as shown in Figure 3-4. Position vector of Q_1 is $\boldsymbol{r_1}$ and position vector of Q_2 is $\boldsymbol{r_2}$ and so on the position vector of Q_n is $\boldsymbol{r_n}$. We want to find out the electric flux density at point P that is caused by all these n number of sources. Position vector of P is $\boldsymbol{r}$.

In order to find out the electric flux density at point P that is caused by all these n number of sources, we apply Superposition Theorem.

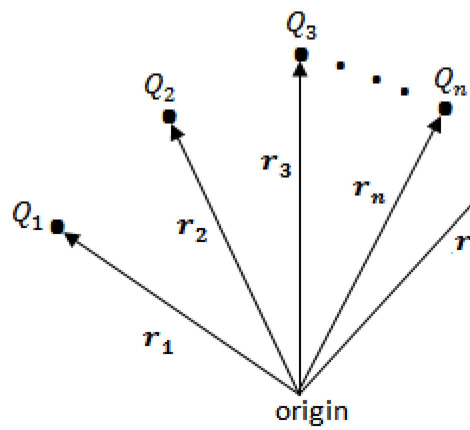


Figure 3-4: n Charge Particles

We assume that there is only one charge that is Q_1 in the vicinity of point P and all the remaining sources do not exist as shown in Figure 3-5. The distance vector extending from Q_1 to P is $\boldsymbol{r}-\boldsymbol{r_1}$.

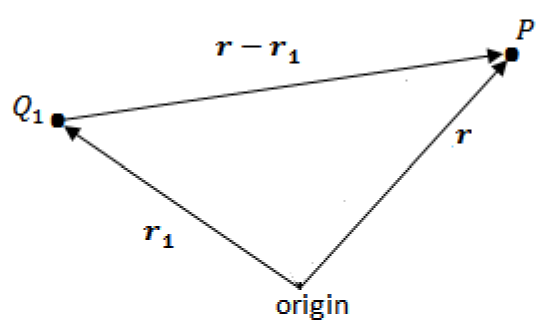


Figure 3-5: Density due to Q_1

Electric flux density due to Q_1 is given by

$$\boldsymbol{D_1} = \frac{Q_1\,(\boldsymbol{r}-\boldsymbol{r_1})}{4\pi\,|\,\boldsymbol{r}-\boldsymbol{r_1}|^3} \tag{3.5}$$

Now, we assume that there is only one charge that is Q_2 in the vicinity of point P and all the remaining sources do not exist as shown in Figure 3-6. The distance vector extending from Q_2 to P is $\boldsymbol{r}-\boldsymbol{r_2}$.

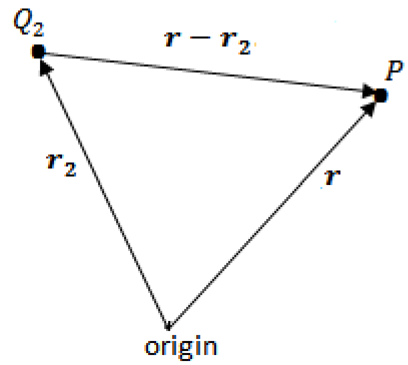


Figure 3-6: Density due to Q_2

Electric flux density due to Q_2 is given by

$$\boldsymbol{D_2} = \frac{Q_2\,(\boldsymbol{r}-\boldsymbol{r_2})}{4\pi\,|\,\boldsymbol{r}-\boldsymbol{r_2}|^3} \tag{3.6}$$

Now, let us assume that there is only one charge that is Q_3 in the vicinity of point P and all the remaining sources do not exist as shown in Figure 3-7. The distance vector extending from Q_3 to P is $\boldsymbol{r}-\boldsymbol{r_3}$.

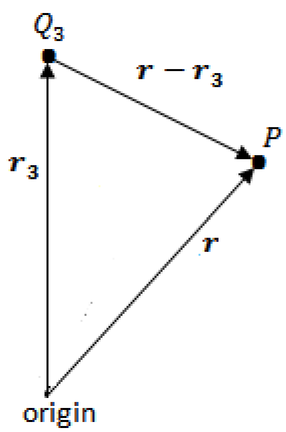


Figure 3-7: Density due to Q_3

Electric flux density due to Q_3 is given by

$$\boldsymbol{D_3} = \frac{Q_3\,(\boldsymbol{r}-\boldsymbol{r_3})}{4\pi\,|\,\boldsymbol{r}-\boldsymbol{r_3}|^3} \tag{3.7}$$

Similarly Electric flux density at point P due to Q_n is given by

$$\boldsymbol{D_n} = \frac{Q_n\,(\boldsymbol{r} - \boldsymbol{r_n})}{4\pi\,|\,\boldsymbol{r} - \boldsymbol{r_n}|^3} \tag{3.8}$$

The vector sum of all these densities results in the total electric flux density at point P.

$$\boldsymbol{D} = \boldsymbol{D_1} + \boldsymbol{D_2} + \boldsymbol{D_3} + \cdots + \boldsymbol{D_n}$$

$$\boldsymbol{D} = \frac{1}{4\pi}\left[\frac{Q_1\,(\boldsymbol{r} - \boldsymbol{r_1})}{|\,\boldsymbol{r} - \boldsymbol{r_1}|^3} + \frac{Q_2\,(\boldsymbol{r} - \boldsymbol{r_2})}{|\,\boldsymbol{r} - \boldsymbol{r_2}|^3} + \frac{Q_3\,(\boldsymbol{r} - \boldsymbol{r_3})}{|\,\boldsymbol{r} - \boldsymbol{r_3}|^3} + \cdots \frac{Q_n\,(\boldsymbol{r} - \boldsymbol{r_n})}{|\,\boldsymbol{r} - \boldsymbol{r_n}|^3}\right] \tag{3.9}$$

The last equation in concise form is given by

$$\boldsymbol{D} = \frac{1}{4\pi}\sum_{i=1}^{n}\frac{Q_i\,(\boldsymbol{r} - \boldsymbol{r_i})}{|\,\boldsymbol{r} - \boldsymbol{r_i}|^3} \tag{3.10}$$

Example 3-1:

Electric flux density in a region is given by

$$\boldsymbol{D} = x^2 z\,\boldsymbol{a_y} + xy^2\,\boldsymbol{a_z} \quad mC/m^2$$

Determine the flux passing through the following surface ia a direction away from the origin.

$$y = 3, \qquad 0 \le x \le 3, \qquad 0 \le z \le 4$$

Solution:

$$\boldsymbol{ds} = dx\,dz\,\boldsymbol{a_y}$$

$$\boldsymbol{D}\,.\,\boldsymbol{ds} = x^2 z\,dx\,dz$$

The flux passing through an open surface is given by

$$\psi = \int \boldsymbol{D}.\boldsymbol{ds}$$

$$\psi = \int_0^3\int_0^4 x^2 z\,dx\,dz$$

$$\psi = \left|\frac{x^3}{3}\right|_0^3 \times \left|\frac{z^2}{2}\right|_0^4$$

$$\psi = 72\, mC$$

Example 3-2:

$100\ mC$ charge is located at the origin, find the flux passing through the surface

$$r = 1, \qquad 0 \le \emptyset \le \pi, \qquad 0 \le \theta \le \pi$$

Solution:

$$\boldsymbol{ds} = r^2 \sin\theta\ d\theta\ \ d\varphi\ \ \boldsymbol{a_r}$$

$$\boldsymbol{D} = \frac{Q}{4\pi r^2}\boldsymbol{a_r}$$

$$\boldsymbol{D} = \frac{100 \times 10^{-3}}{4\pi r^2}\boldsymbol{a_r}$$

$$\boldsymbol{D.ds} = \frac{100 \times 10^{-3}}{4\pi}\sin\theta\ d\theta\ \ d\varphi$$

The flux passing through an open surface is given by

$$\psi = \int \boldsymbol{D.ds}$$

$$\psi = \frac{100 \times 10^{-3}}{4\pi}\int_0^\pi \sin\theta\ \ d\theta\ \int_0^\pi d\emptyset$$

$$\psi = \frac{100 \times 10^{-3}}{4\pi} \times 2\pi$$

$$\psi = 50\, mC$$

3-5 Electric Flux Density due to a Line Charge

Assume that charge is uniformly distributed along the length of a line as shown in Figure 3-8. Differential charge on the differential portion of the line is

$$dQ = \rho_L d\ell \tag{3.11}$$

In order to find out the Electric flux density at point P that is caused by the line charge, we consider small portion of the source first and determine the density at P that is caused by the small portion. Position vector of the small portion of the source is $\boldsymbol{r_1}$and position vector of Point P is $\boldsymbol{r}$**.** The distance vector extending from dQ to P is $(\boldsymbol{r}-\boldsymbol{r_1})$**.** Differential electric flux density at point P that is caused by the differential charge is given by

$$\boldsymbol{dD} = \frac{dQ\ (\boldsymbol{r}-\boldsymbol{r_1})}{4\pi\,|\,\boldsymbol{r}-\boldsymbol{r_1}|^3} \tag{3.12}$$

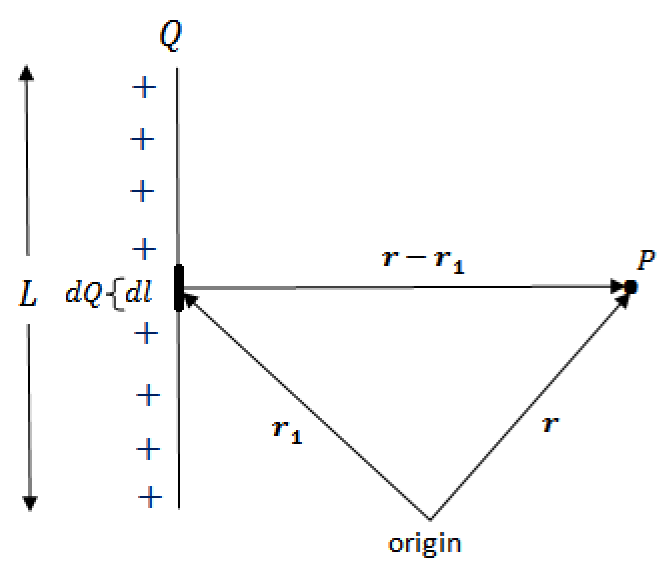


Figure 3-8: Electric Flux Density due to Line Charge

Putting the value of dQ in equation 3.12, we obtain

$$dD = \frac{\rho_L d\ell \,(\boldsymbol{r} - \boldsymbol{r_1})}{4\pi \,|\, \boldsymbol{r} - \boldsymbol{r_1}|^3} \tag{3.13}$$

Total density that is caused by the entire source

$$\boldsymbol{D} = \int \frac{\rho_L d\ell \,(\boldsymbol{r} - \boldsymbol{r_1})}{4\pi \,|\, \boldsymbol{r} - \boldsymbol{r_1}|^3} \tag{3.14}$$

3-6 Electric Flux Density due to Surface Charge

Assume that charge is uniformly distributed along surface of a sheet as shown in Figure 3-9. Total charge on surface S of the sheet is Q coulomb. Charge per unit area is known as surface charge density which is represented by ρ_S. Differential charge on the differential portion of the surface is

$$dQ = \rho_S \, ds \tag{3.15}$$

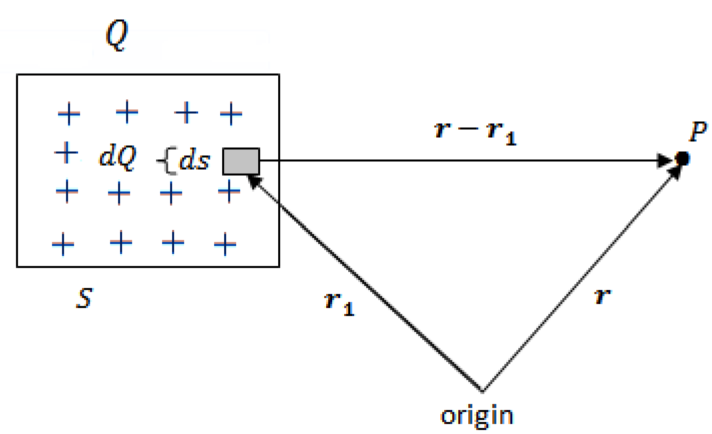


Figure 3-9: Density due to Surface Charge

In order to find out the electric flux density at point P that is caused by the surface charge, we consider small portion of the source first and determine the density at P that is caused by the small portion. Position vector of the small portion of the source is $\boldsymbol{r_1}$ and position vector of Point P is $\boldsymbol{r}$. The distance vector extending from dQ to P is $(\boldsymbol{r} - \boldsymbol{r_1})$. Differential electric flux density at point P that is caused by the differential charge is given by

$$\boldsymbol{dD} = \frac{dQ\,(\boldsymbol{r} - \boldsymbol{r_1})}{4\pi\,|\,\boldsymbol{r} - \boldsymbol{r_1}|^3} \tag{3.16}$$

Putting the value of dQ in equation 3.16, we obtain

$$\boldsymbol{dD} = \frac{\rho_S\,ds\,(\boldsymbol{r} - \boldsymbol{r_1})}{4\pi\,|\,\boldsymbol{r} - \boldsymbol{r_1}|^3} \tag{3.17}$$

Total density that is caused by the entire source

$$\boldsymbol{D} = \int \frac{\rho_S\,ds\,(\boldsymbol{r} - \boldsymbol{r_1})}{4\pi\,|\,\boldsymbol{r} - \boldsymbol{r_1}|^3} \tag{3.18}$$

3-7 Electric Flux Density due to Volume Charge

Assume that charge is uniformly distributed inside a volume V as shown in Figure 3-10. Total charge inside the given volume is Q coulomb. Charge per unit volume is known as volume charge density which is represented by ρ_v.

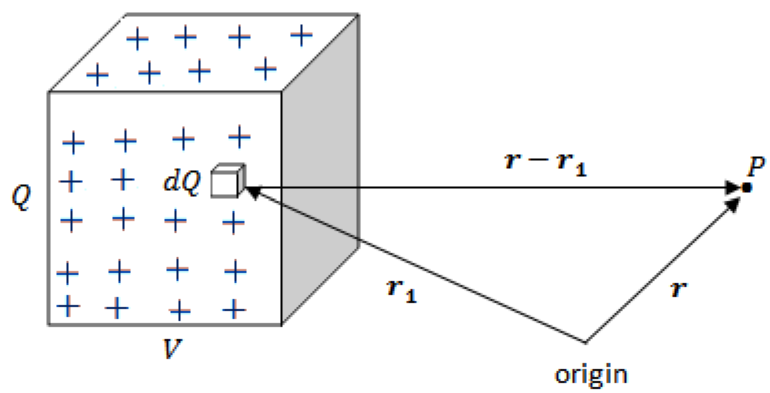


Figure 3-10: Density due to Volume Charge

Differential charge in the differential portion of the volume is

$$dQ = \rho_v\,dv \tag{3.19}$$

In order to find out the electric flux density at point P that is caused by the volume charge, we consider small portion of the source first and determine the intensity at P that is caused by the small portion. Position vector of the small portion of the source is $\boldsymbol{r_1}$ and position vector of Point P is $\boldsymbol{r}$. The distance vector extending from dQ to P is $(\boldsymbol{r}-\boldsymbol{r_1})$. Differential electric flux density at point P that is caused by the differential charge is given by

$$\boldsymbol{dD} = \frac{dQ\,(\boldsymbol{r}-\boldsymbol{r_1})}{4\pi\,|\,\boldsymbol{r}-\boldsymbol{r_1}|^3} \tag{3.20}$$

Putting the value of dQ in equation 3.19, we obtain

$$\boldsymbol{dD} = \frac{\rho_v\,dv\,(\boldsymbol{r}-\boldsymbol{r_1})}{4\pi\,|\,\boldsymbol{r}-\boldsymbol{r_1}|^3} \tag{3.21}$$

Total density that is caused by the entire source

$$\boldsymbol{D} = \int \frac{\rho_v\,dv\,(\boldsymbol{r}-\boldsymbol{r_1})}{4\pi\,|\,\boldsymbol{r}-\boldsymbol{r_1}|^3} \tag{3.22}$$

3-8 Electric Flux Density due to Infinite Line Charge

Consider an infinite line charge extending from $-\infty$ to ∞ along $z-axis$ as shown in Figure 3-11. Line charge density of the source is ρ_L. Length of the small portion of the Source is $d\ell$. Position vector of the small portion of the source is $\boldsymbol{r_1}$ and position vector of Point P is $\boldsymbol{r}$. The distance vector extending from $d\ell$ to P is $(\boldsymbol{r}-\boldsymbol{r_1})$.

Electric field intensity at point P that is caused by a line charge is given by

$$\boldsymbol{E} = \frac{\rho_L}{2\pi\varepsilon\rho}\boldsymbol{a_\rho} \tag{3.23}$$

as $$\boldsymbol{D} = \varepsilon\boldsymbol{E}$$

Electric flux density at point P is given by

$$\boldsymbol{D} = \frac{\rho_L}{2\pi\rho}\boldsymbol{a_\rho} \tag{3.24}$$

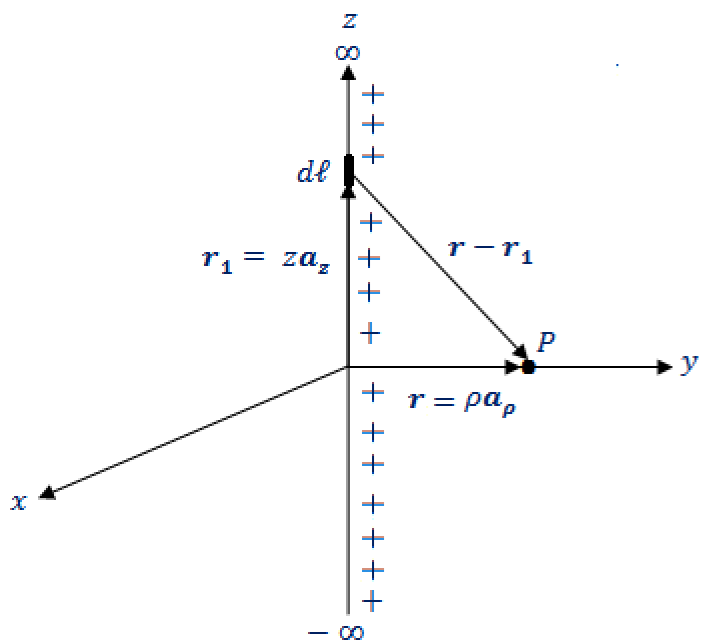


Figure 3-11: Electric Flux Density due to Infinite Line Charge

If we enclose the source in a cylinder as shown in Figure 3-12, electric flux density on the surface of the cylinder will be constant. Similarly if we consider a line parallel to the source then electric flux density along this line will be constant.

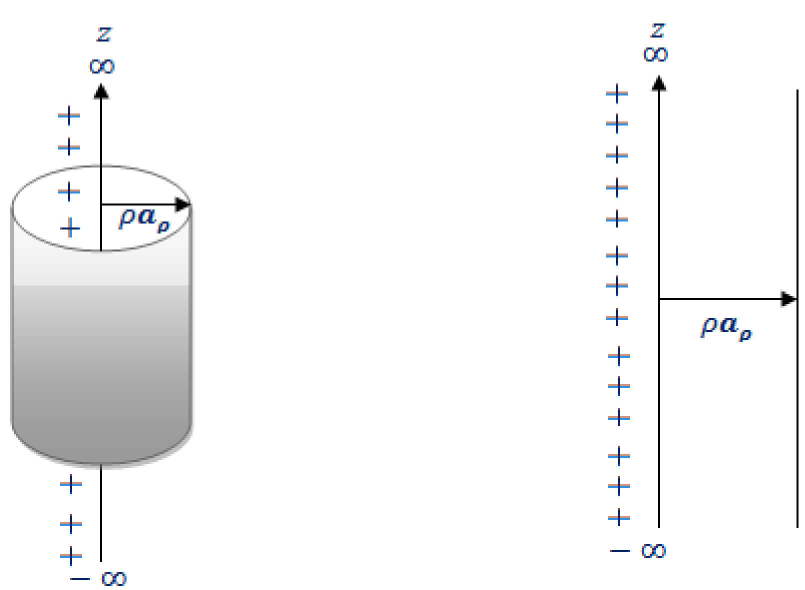


Figure 3-12: Density on the Surface of a Cylinder and along a line parallel to Source

3-9 Electric Flux Density due to Infinite Sheet of Charge

Consider an infinite sheet of charge which is located in $x = 0$ plane as shown in Figure 3-13. We want to find out electric flux density at point P. As electric field intensity at point P that is caused by a line charge is given by

$$\boldsymbol{E} = \frac{\rho_s}{2\varepsilon}\boldsymbol{a_x} \tag{3.25}$$

Therefore

$$\boldsymbol{D} = \frac{\rho_s}{2}\boldsymbol{a_x} \tag{3.26}$$

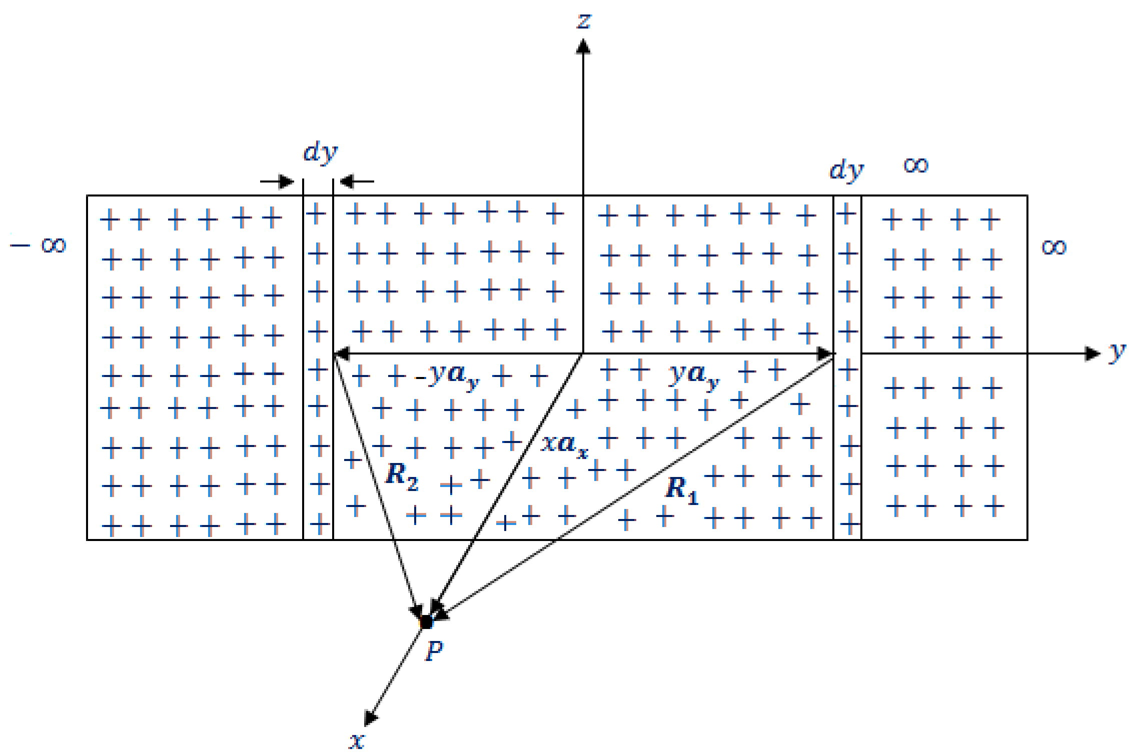


Figure 3-13: Electric Flux Density due to Infinite Sheet of Charge

Where $\boldsymbol{a_x}$ is a unit vector normal to the sheet. Let us represent this unit vector normal to the sheet by $\boldsymbol{a_n}$. Equation no 3.26 in generic form can be written as

$$\boldsymbol{D} = \frac{\rho_s}{2}\boldsymbol{a_n} \tag{3.27}$$

3-10 Gauss's Law

Consider a closed surface in the form of a sphere of radius r as shown in Figure 3-14. A point charge of Q coulomb is located at the center of the sphere. We consider a point P on the small portion $\boldsymbol{ds}$ of the sphere and the electric flux density at this point is given by

$$\boldsymbol{D} = \frac{Q}{4\pi r^2}\boldsymbol{a_r} \tag{3.28}$$

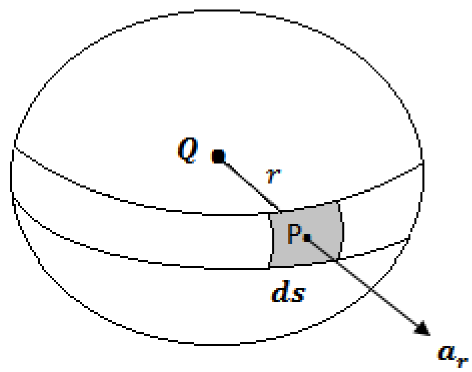


Figure 3-14: Charge of Q coulomb in a Sphere of Radius r

The electric flux passing through the small portion of the sphere in the outward direction is given by

$$d\psi = \boldsymbol{D}.\boldsymbol{ds} \tag{3.29}$$

Consider the small portion $\boldsymbol{ds}$ of the sphere as shown in Figure 3-15, the mathematical expression for $\boldsymbol{ds}$ is given by

$$\boldsymbol{ds} = r^2 sin\theta d\theta d\emptyset \boldsymbol{a_r}$$

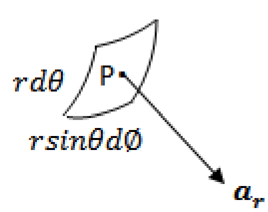


Figure 3-15: Small porion of the Sphere

The electric flux passing through the differential portion of the surface is calculated as

$$d\psi = \frac{Q}{4\pi r^2}\boldsymbol{a_r}.\, r^2 sin\theta d\theta d\emptyset \boldsymbol{a_r}$$

$$d\psi = \frac{Q}{4\pi} sin\theta d\theta d\emptyset \qquad (3.30)$$

The electric flux passing through the closed surface in the outward direction is calculated as

$$\psi = \oint \boldsymbol{D.ds}$$

$$\psi = \int_0^{\pi}\int_0^{2\pi} \frac{Q}{4\pi} sin\theta d\theta d\emptyset \qquad (3.31)$$

$$\psi = \frac{Q}{4\pi}\int_0^{\pi} sin\theta d\theta \int_0^{2\pi} d\emptyset$$

$$\psi = \frac{Q}{4\pi}(-\cos\theta)_0^{\pi} \times (\emptyset)_0^{2\pi}$$

$$= \frac{Q}{4\pi}(2)(2\pi)$$

$$\psi = \oint \boldsymbol{D.ds} = Q \qquad (3.32)$$

As the charge of Q coulomb is enclosed by the closed surface, therefore

$$\psi = \oint \boldsymbol{D.ds} = Q_{enclosed} \qquad (3.33)$$

This is the mathematical model of Gauss's Law, which states that electric flux passing through a closed surface is equal the charge enclosed by the closed surface. Let us consider this charge of Q coulomb in a very small sphere as shown in Figure 3-16.

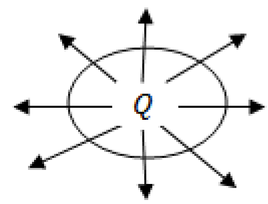


Figure 3-16: Charge of Q coulomb in a Closed Surface

The total flux generated by the source of Q coulomb will pass through this closed surface in the outward direction. That is

$$\psi_t = \oint \boldsymbol{D.ds} = Q$$

So we conclude that the total flux generated by a source is equal to the charge on the source.

Example 3-3:

Volume charge density of

$$\rho_v = 4x\, z^2 \cos\frac{y}{2} \quad C/m^3$$

is located in a region defined by

$$0 \le y \le 3.14, \qquad 0 \le x \le 1, \qquad 0 \le z \le 3$$

Determine the electric flux passing through this surface in the outwared direction.

Solution:

The surface in a closed rectangular suface. According to Gauss's law, the flux passing through a closed surface in the outward direction is equal to the charge enclosed by the closed surface.

$$\psi = \oint \boldsymbol{D.ds} = Q_{enclosed}$$

$$dv = dx\, dy\, dz$$

$$Q_{enclosed} = \int \rho_v dv$$

Therefore the flux passing through this closed surface is given by

$$Q = 4\int_0^1 xdx \int_0^3 z^2dz \int_0^{3.14} \cos\frac{y}{2}\,dy$$

$$\psi = 4\left|\frac{x^2}{2}\right|_0^1 \times \left|\frac{z^3}{3}\right|_0^3 \times 2\left|\sin\frac{y}{2}\right|_0^{3.14}$$

$$\psi = 4 \times \frac{1}{2} \times \frac{27}{3} \times 2 \times 1$$

$$\psi = 36\ C$$

Example 3-4:

Volume charge density of

$$\rho_v = 4\rho\, z^3\ \ mC/m^3$$

is located in a region defined by

$$0 \le \rho \le 3, \qquad 0 \le \emptyset \le 2\pi, \qquad 0 \le z \le 4$$

Determine the electric flux passing through this surface in the outwared direction.

Solution:

The surface in a closed cylendrical suface. According to Gauss's law, the flux passing through a closed surface in the outward direction is equal to the charge enclosed by the closed surface.

$$\psi = \oint \boldsymbol{D}.\boldsymbol{ds} = Q_{enclosed}$$

$$dv = \rho d\rho\, d\emptyset\, dz$$

$$Q_{enclosed} = \int \rho_v dv$$

Therefore the flux passing through this closed surface is given by

$$Q = 4 \times 10^{-3} \int_0^3 \rho^2 d\rho \int_0^4 z^3 dz \int_0^{2\pi} d\emptyset$$

$$\psi = 4 \times 10^{-3} \left| \frac{\rho^3}{3} \right|_0^3 \times \left| \frac{z^4}{4} \right|_0^3 \times | \emptyset |_0^{2\pi}$$

$$\psi = 4 \times 10^{-3} \times \frac{27}{3} \times 64 \times 2\pi$$

$$\psi = 14.4\ C$$

Example 3-5:

Volume charge density of

$$\rho_v = r \cos^2 \emptyset \quad mC/m^3$$

is located in a region defined by

$$0 \leq r \leq 4, \qquad 0 \leq \emptyset \leq 2\pi, \qquad 0 \leq \theta \leq \pi$$

Determine the electric flux passing through this surface in the outwared direction.

Solution:

The surface in a closed spherical suface. According to Gauss's law, the flux passing through a closed surface in the outward direction is equal to the charge enclosed by the closed surface.

$$\psi = \oint \boldsymbol{D} . \boldsymbol{ds} = Q_{enclosed}$$

$$dv = r^2 \sin\theta \, d\theta \, dr \, d\varphi$$

$$Q_{enclosed} = \int \rho_v dv$$

Therefore the flux passing through this closed surface is given by

$$Q = 10^{-3} \int_0^4 r^3 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} \cos^2 \emptyset \, d\emptyset$$

$$\psi = 10^{-3} \left| \frac{r^4}{4} \right|_0^4 \times |-\cos\theta|_0^\pi \times \left| \frac{\emptyset}{2} \right|_0^{2\pi}$$

$$\psi = 10^{-3} \times 64 \times 2 \times \pi$$

$$\psi = 0.402\ C$$

3-11 Applications of Gauss's Law

3-11-1 Coaxial Cable

A coaxial cable consists of two conductors, the inner conductor of radius a and the outer conductor of radius b as shown in Figure 3-17. There is a dielectric material in between the two conductors that has a dielectric constant of $\mathcal{E}_r$. Positive charge is uniformly distributed on the outer surface of the inner conductor of the coaxial cable. The surface charge density of the inner conductor is represented by $\rho_{S_{in}}$. According to electrostatic induction, the positive charge on the outer surface of the inner conductor will induce the same amount of negative charge on the inner surface of the outer conductor.
Surface area of length L of the inner conductor can be calculated with the help of Figure 3-18. Surface area of length L of the inner conductor

$$S_{in} = 2\pi a L$$

The charge on length L of the inner conductor

$$Q_{in} = \rho_{S_{in}} \times 2\pi a L \tag{3.34}$$

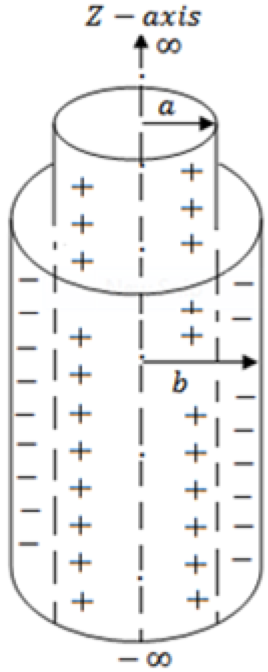


Figure 3-17: Infinite Coaxial Cable

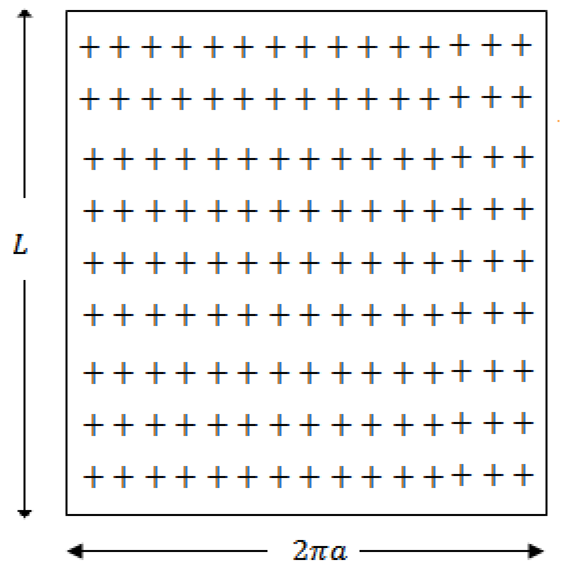


Figure 3-18: Surface Area of Inner Conductor

Surface area of length L of the outer conductor can be calculated with the help of Figure 3-19.

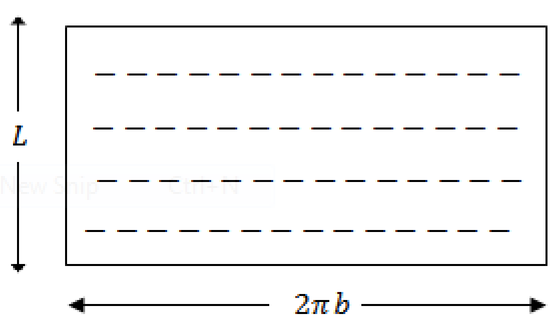


Figure 3-19: Surface Area of Outer Conductor

Surface area of length L of the outer conductor

$$S_{out} = 2\pi bL$$

The charge on length L of the outer conductor

$$Q_{out} = \rho_{S_{out}} \times 2\pi bL \tag{3.35}$$

According to electrostatic induction Phenomenon

$$Q_{out} = -Q_{in} \tag{3.36}$$

$$\rho_{S_{out}} \times 2\pi bL = -\rho_{S_{in}} \times 2\pi aL \tag{3.37}$$

$$\rho_{S_{out}} = \frac{-\rho_{S_{in}} \times a}{b} \tag{3.38}$$

We are going to find out the electric field intensity at different location of coaxial cable. In case 1, we consider a point in the dielectric material of the coaxial cable as shown in Figure 3-20.

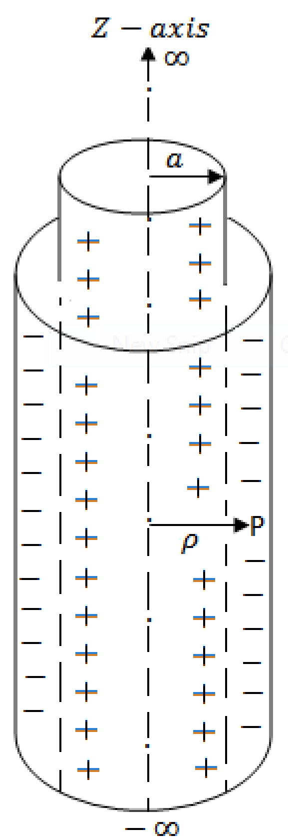


Figure 3-20: Point in the Dielectric Material of the Coaxial Cable

Case 1: $a < \rho < b$

In order to determine the intensity at P, we apply Gauss's law. For the application of Gauss's law consider a closed surface in the form of a cylinder of radius ρ. The top view of this scenario is shown in Figure 3-21.

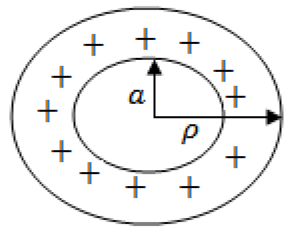


Figure 3-21: Top View of Inner Conductor inside the Gaussian Cylinder

Obviously the entire inner conductor is located inside the closed Gaussian cylinder; we assume that the radius of the inner conductor is so small that it almost behaves like an infinite line charge. The electric field intensity due to an infinite line charge is given by

$$\boldsymbol{E} = \frac{\rho_L}{2\pi\varepsilon\rho}\boldsymbol{a_\rho} \tag{3.39}$$

Where ρ_L of the inner conductor is computed from equation 3.34 as

$$\frac{Q_{in}}{L} = \rho_{S_{in}} \times 2\pi a \tag{3.40}$$

$$\boldsymbol{E} = \frac{\rho_{S_{in}} \times 2\pi a}{2\pi\varepsilon\rho}\boldsymbol{a_\rho}$$

$$\boldsymbol{E} = \frac{\rho_{S_{in}} \times a}{\varepsilon\rho}\boldsymbol{a_\rho} \tag{3.41}$$

Electric flux density at the point under observation is given by

$$\boldsymbol{D} = \frac{\rho_{S_{in}} \times a}{\rho}\boldsymbol{a_\rho} \tag{3.42}$$

In case 2, we consider a point outside the coaxial cable as shown in Figure 3-22.

Case 2: $\rho > b$

In order to determine the intensity at P, we apply Gauss's law. For the application of Gauss's law consider a closed surface in the form of a cylinder of radius ρ. The top view of this scenario is shown in Figure 3-23.Obviously the entire coaxial cable is located inside the closed Gaussian cylinder. We apply the Gauss's law over here, which states that electric flux passing through a closed surface is equal the charge enclosed by the closed surface.

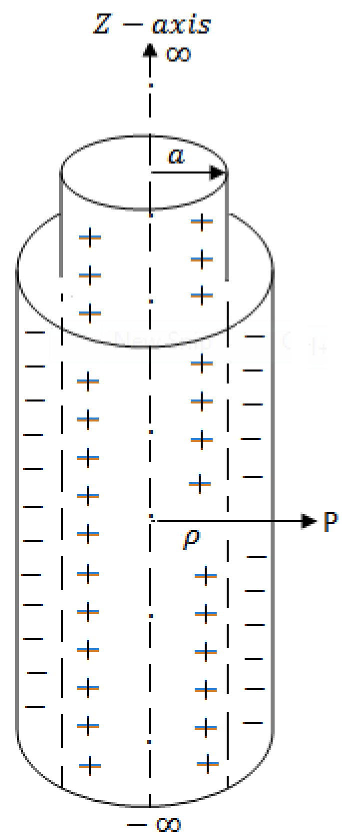


Figure 3-22: Point outside of the Coaxial Cable

$$\psi = \oint \boldsymbol{D.ds} = Q_{enclosed} \tag{3.43}$$

$$As \;\; Q_{enclosed} = Q + (-Q) = 0$$

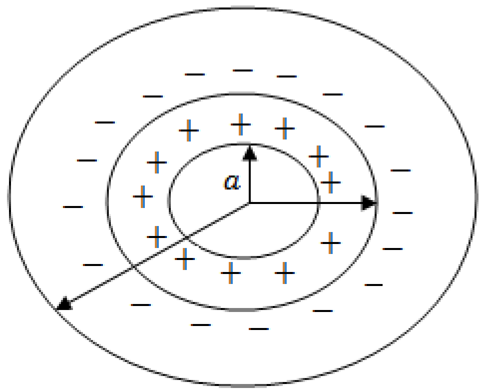


Figure 3-23: Top View of Coaxial Cable inside the Gaussian Cylinder

$$So \qquad \psi = \oint \boldsymbol{D}.\boldsymbol{ds} = 0 \qquad (3.44)$$

Therefore $\boldsymbol{D} = \boldsymbol{E} = 0$

3-11-2 Spherical Charge

The Charge is uniformly distributed on the outer surface of a sphere of radius a as shown in Figure 3-24. We want to find out the electric field intensity inside and outside the source.

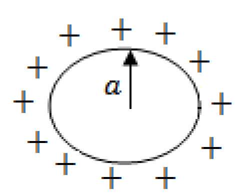


Figure 3-24: Spherical Charge

Surface area of the sphere is given by

$$S = 4\pi a^2$$

Charge on the surface of the sphere is given by

$$Q = \rho_s \times 4\pi a^2 \qquad (3.45)$$

Consider a point outside the source, the radial distance between the center of the source and point P is represented by r. In order to determine the intensity at P, we apply Gauss's law. For the application of Gauss's law consider a closed surface in the form of a sphere of radius r as shown in Figure 3-25.

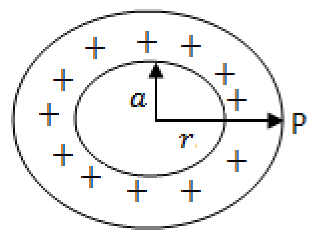


Figure 3-25: Point outside the Spherical Charge

The entire source is located inside the Gaussian surface. We assume that the radius of the source is so small that it almost behaves like a point charge.

The electric flux density due to point charge is given by

$$\boldsymbol{D} = \frac{Q}{4\pi r^2}\boldsymbol{a_r} \tag{3.46}$$

Putting the value of Q in equation 3.46, we obtain

$$\boldsymbol{D} = \frac{\rho_s \times 4\pi a^2}{4\pi r^2}\boldsymbol{a_r} \tag{3.47}$$

$$\boldsymbol{D} = \frac{\rho_s \times a^2}{r^2}\boldsymbol{a_r} \tag{3.48}$$

Consider a point inside the source, the radial distance between the center of the source and point P is represented by r. In order to determine the intensity at P, we apply Gauss's law. For the application of Gauss's law consider a closed surface in the form of a sphere of radius r as shown in Figure 3-26.

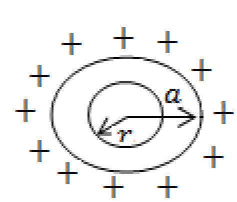


Figure 3-26: Point inside the Spherical Charge

The source is located outside the Gaussian surface. No charge has been enclosed by the Gaussian surface.

$$\psi = \oint \boldsymbol{D}.\boldsymbol{ds} = Q_{enclosed} \tag{3.49}$$

$$As \;\; Q_{enclosed} = 0$$

$$So \qquad \psi = \oint \boldsymbol{D}.\boldsymbol{ds} = 0 \tag{3.50}$$

Therefore $\boldsymbol{D} = \boldsymbol{E} = 0$

Example 3.6:
Surface charge density of $100\ \mu C/m^2$ is located at $r = 2$. Find $\boldsymbol{D}$ and $\boldsymbol{E}$ at $r = 4$ in free space.

Solution:

$$\boldsymbol{D} = \frac{\rho_s \times a^2}{r^2}\boldsymbol{a_r}$$

$$\boldsymbol{D} = \frac{100 \times 10^{-6} \times 4}{16}\boldsymbol{a_r}$$

$$\boldsymbol{D} = 25\ \boldsymbol{a_r}\ \ \mu C/m^2$$

$$\boldsymbol{E} = \frac{25 \times 10^{-6} \times 10^{12}}{8.85}\boldsymbol{a_r}$$

$$\boldsymbol{E} = 2.82 \times\ 10^6 \boldsymbol{a_r}\ \ V/m$$

3-12 Maxwell's First Equation

A source in the form of volume charge is enclosed in a Gaussian surface as shown in Figure 3-27. The total charge of the source is given by

$$Q = \int \rho_v\ dv \tag{3.51}$$

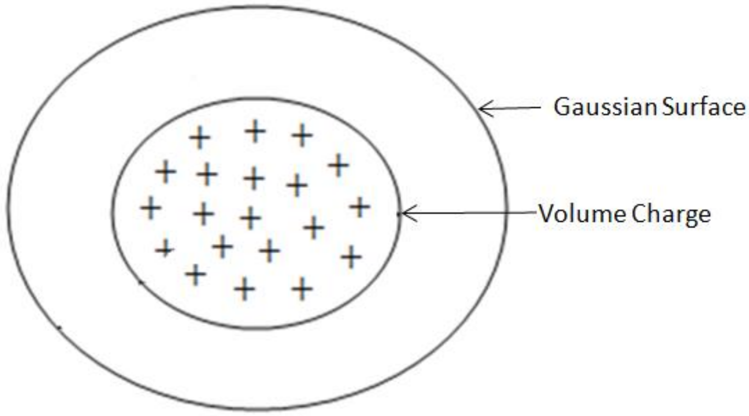


Figure 3-27: Volume Charge inside a Gaussian Surface

The volume charge is located inside the Gaussian surface. We apply the Gauss's law over here, which states that electric flux passing through this closed surface is equal the charge enclosed by the closed surface.

$$\psi = \oint \boldsymbol{D.ds} = Q_{enclosed} \qquad (3.52)$$

Evidently, equation no 3.52 can be written as

$$\oint \boldsymbol{D.ds} = \int \rho_v \; dv \qquad (3.53)$$

The Divergence Theorem states that

$$\oint \boldsymbol{D.ds} = \int (\boldsymbol{\nabla.D}) \; dv \qquad (3.54)$$

Where $(\boldsymbol{\nabla.D})$ stands for the divergence of Vector $\boldsymbol{D}$.

Divergence of vector $\boldsymbol{D}$ in Rectangular Coordinate System is given by

$$\boldsymbol{\nabla.D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \qquad (3.55)$$

Divergence of vector $\boldsymbol{D}$ in Cylindrical Coordinate System is given by

$$\nabla . \boldsymbol{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho D_{\rho}\right) + \frac{1}{\rho}\frac{\partial D_{\emptyset}}{\partial \emptyset} + \frac{\partial D_z}{\partial z} \tag{3.56}$$

Divergence of vector $\boldsymbol{D}$ in Spherical Coordinate System is given by

$$\nabla . \boldsymbol{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{rsin\theta}\frac{\partial D_{\emptyset}}{\partial \emptyset} + \frac{1}{rsin\theta}\left(sin\theta \frac{\partial D_{\theta}}{\partial \theta}\right) \tag{3.57}$$

Equation no 3.52 can be written as

$$\int (\nabla . \boldsymbol{D})\ dv = \int \rho_v\ dv \tag{3.58}$$

$$\nabla . \boldsymbol{D} = \rho_v \tag{3.59}$$

The last equation is known as 1st equation of Maxwell.

Example 3-7:

If the electric flux density is given by

$$\boldsymbol{D} = x^2 y\, z\, \boldsymbol{a_x} + x^2\, z\, \boldsymbol{a_y} + y\, x\, \boldsymbol{a_z}\ mC/m^2$$

Determine ρ_v (b) if this ρ_v is located inside the closed surface defined by

$$0 \le x \le 2, 0 \le y \le 2,\ 0 \le z \le 2$$

Then determine the flux passing this surface.

Solution:

$$\nabla . \boldsymbol{D} = \rho_v$$

$$\nabla . \boldsymbol{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\rho_v = 2xyz\ \ mC/m^3$$

(b)

$$dv = dx\, dy\, dz$$

$$\psi = \int (\boldsymbol{\nabla}.\boldsymbol{D})\ dv \ \ = \ \int \rho_v\ dv$$

$$\psi = 10^{-3} \int_0^2 2x dx \int_0^2 y dy \int_0^2 z\ dz$$

$$\psi = 2 \times 10^{-3} \left| \frac{x^2}{2} \right|_0^2 \times \left| \frac{y^2}{2} \right|_0^2 \times \left| \frac{z^2}{2} \right|_0^2$$

$$\psi = 16 \times 10^{-3}\ C$$

Chapter 4

Energy and Voltage

4-1 Electrical Energy

Consider a source of Q_1 coulomb that creates an electric field intensity of $\boldsymbol{E}\ v/m$ as shown in Figure 4-1. There will be a force of repulsion on the charge of Q coulomb and it will move from the initial point A to the final point B in the direction of field.

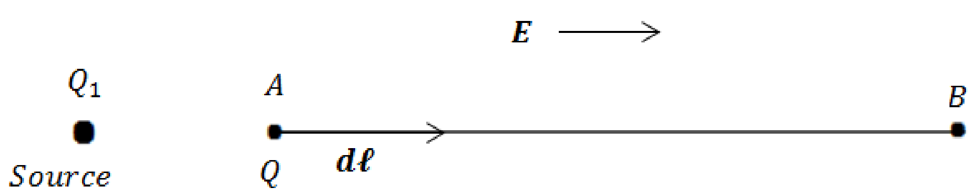


Figure 4-1: Electric Field Generated by Source Q_1

In order to find out the energy that is supplied by source Q_1 to move this charge of Q coulomb from the initial point A to the final point B, we consider a very small portion of the distance in the direction of intensity

$$\boldsymbol{d\ell} = d\ell\ \boldsymbol{a_r}$$

The electric force and electric field intensity are given by

$$\boldsymbol{F} = F\ \boldsymbol{a_r}$$

$$\boldsymbol{E} = E\ \boldsymbol{a_r}$$

The differential energy supplied by the source to move the charge of Q coulomb along the differential distance $\boldsymbol{d\ell}$ is given by

$$dw = F d\ell \qquad (4.1)$$

Equation 4.1 can be written as

$$dw = \boldsymbol{F}.\boldsymbol{d\ell} \qquad (4.2)$$

The electric force on the charge of Q coulomb is given by

$$\boldsymbol{F} = Q\boldsymbol{E}$$

Therefore

$$dw = Q\boldsymbol{E}.\boldsymbol{d\ell} \tag{4.3}$$

The total energy supplied by the source to move the charge of Q coulomb from the initial point A to the final point B is given by

$$W = Q\int_A^B \boldsymbol{E}.\boldsymbol{d\ell} \tag{4.4}$$

We need an external source of energy if we want to move the charge of Q coulomb from the initial point B to the final point A against the field of the source as shown in Figure 4-2.

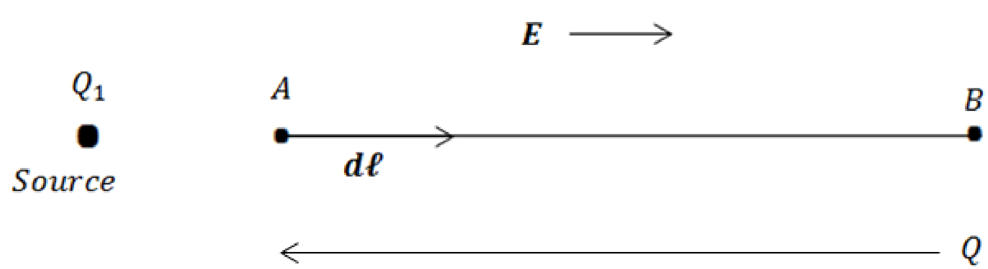


Figure 4-2: Moving a Charge of Q coulomb against the Field

The differential energy that is required to move this charge of Q coulomb along the differential distance $\boldsymbol{d\ell}$ against the field is given by

$$dw = -\boldsymbol{F}.\boldsymbol{d\ell} \tag{4.5}$$

$$dw = -Q\boldsymbol{E}.\boldsymbol{d\ell} \tag{4.6}$$

The total energy which is required to move the charge of Q coulomb from the initial position B to the final position A against the field is given by

$$W = -Q\int_B^A \boldsymbol{E}.\boldsymbol{d\ell} \qquad (4.7)$$

4-2 Line Integral

Energy will be required to move a charge of Q coulomb from the initial point B to the final point A in a uniform electric field $\boldsymbol{E}$ as shown in Figure 4-3. To determine this energy, we divide the entire path into a very large number of very small segments. These segments are represented by $\Delta\boldsymbol{L_1}$, $\Delta\boldsymbol{L_2}$, $\Delta\boldsymbol{L_3}$, $\Delta\boldsymbol{L_4}$, $\Delta\boldsymbol{L_5}$, $\Delta\boldsymbol{L_6}$ and $\Delta\boldsymbol{L_7}$.

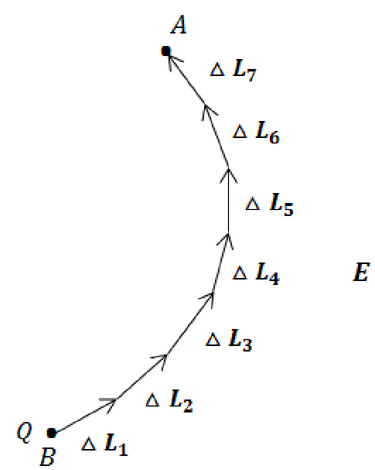


Figure 4-3: Moving a Charge of Q Coulomb in a Uniform Field

The energy that is required to move this charge of Q coulomb along the small segment $\Delta\boldsymbol{L_1}$ is given by

$$\Delta W_1 = -Q\boldsymbol{E}.\Delta\boldsymbol{L_1} \qquad (4.8)$$

The energy that is required to move this charge of Q coulomb along the small segment $\Delta\boldsymbol{L_2}$ is given by

$$\Delta W_2 = -Q\boldsymbol{E}.\Delta\boldsymbol{L_2} \qquad (4.9)$$

The energy that is required to move this charge of Q coulomb along the small segment $\Delta\boldsymbol{L_3}$ is given by

$$\Delta W_3 = -Q\boldsymbol{E}.\Delta\boldsymbol{L_3} \qquad (4.10)$$

And so on, the energy that is required to move this charge of Q coulomb along the small segment $\Delta \boldsymbol{L_7}$ is given by

$$\Delta W_7 = -Q\boldsymbol{E}.\Delta \boldsymbol{L_7} \tag{4.11}$$

The total energy will be equal to the sum of all these energies, that is

$$W = \Delta W_1 + \Delta W_2 + \Delta W_3 + \Delta W_4 + \Delta W_5 + \Delta W_6 + \Delta W_7 \tag{4.12}$$

$$W = -Q\boldsymbol{E}.\Delta \boldsymbol{L_1} - Q\boldsymbol{E}.\Delta \boldsymbol{L_2} - Q\boldsymbol{E}.\Delta \boldsymbol{L_3} - \cdots - Q\boldsymbol{E}.\Delta \boldsymbol{L_7} \tag{4.13}$$

$$W = -Q\boldsymbol{E}.(\Delta \boldsymbol{L_1} + \Delta \boldsymbol{L_2} + \Delta \boldsymbol{L_3} + \Delta \boldsymbol{L_4} + \Delta \boldsymbol{L_5} + \Delta \boldsymbol{L_6} + \Delta \boldsymbol{L_7}) \tag{4.14}$$

As the vector sum of $\Delta \boldsymbol{L_1}$, $\Delta \boldsymbol{L_2}$, $\Delta \boldsymbol{L_3}$, $\Delta \boldsymbol{L_4}$, $\Delta \boldsymbol{L_5}$, $\Delta \boldsymbol{L_6}$ and $\Delta \boldsymbol{L_7}$ results in $\boldsymbol{L_{BA}}$ as shown in Figure 4-4.

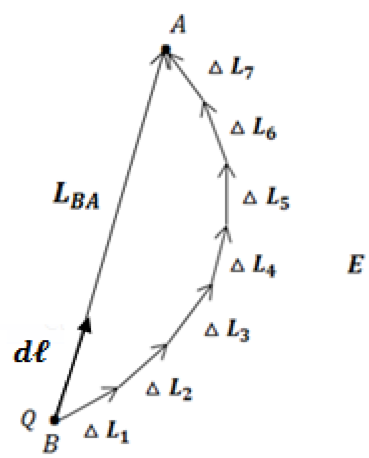


Figure 4-4: Displacement Vector between Points A and B

$$\Delta \boldsymbol{L_1} + \Delta \boldsymbol{L_2} + \Delta \boldsymbol{L_3} + \Delta \boldsymbol{L_4} + \Delta \boldsymbol{L_5} + \Delta \boldsymbol{L_6} + \Delta \boldsymbol{L_7} = \boldsymbol{L_{BA}}$$

$$W = -Q\boldsymbol{E}.\boldsymbol{L_{BA}} \tag{4.15}$$

We conclude that we can choose any path between the two points to move the charge of Q coulomb from the initial position B to the final position A as the same amount of energy will be expended. This energy can be calculated with the help of following equation as well

$$W = -Q\int_B^A \boldsymbol{E}.\boldsymbol{d\ell} \tag{4.16}$$

As electric field intensity is uniform, therefore

$$W = -Q\boldsymbol{E}.\int_B^A \boldsymbol{d\ell} \tag{4.17}$$

As

$$\int_B^A \boldsymbol{d\ell} = \boldsymbol{L_{BA}}$$

Therefore

$$W = -Q\boldsymbol{E} \, . \, \boldsymbol{L_{BA}} \tag{4.18}$$

Example 4-1:

Find the energy that is required to move a charge of $2mC$ from $Q(6,\ 8, 0) to\ P(0,\ 0, 0)$ against the electric field intensity $\boldsymbol{E} = 2x\boldsymbol{a_x} + 4y\boldsymbol{a_y}\ V/m$.

Solution:

The charge is moved along the straight segments from point $Q(6,\ 8,\ 0)$ to $P(0,\ 0,\ 0)$ as shown in Figure 4-5.

Consider Path C_1 :

$y = 8\,,\quad dy = 0$

$z = 0\,,\quad dz = 0$

$0 \leq x\ \leq 6$

$\boldsymbol{d\ell} = dx\boldsymbol{a_x} + dy\boldsymbol{a_y} + dz\boldsymbol{a_z}$

$\boldsymbol{d\ell} = dx\boldsymbol{a_x}$

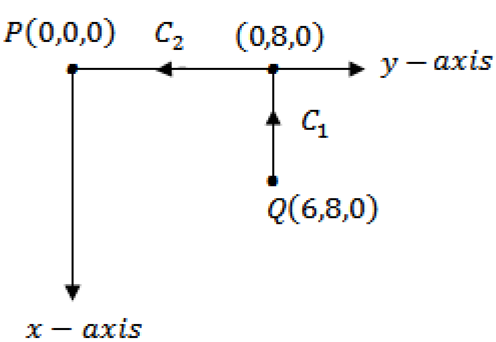


Figure 4.5: Straight Segments

$$\boldsymbol{E}.\boldsymbol{d\ell} = 2xdx$$

Using

$$W_1 = -Q\int \boldsymbol{E}.\boldsymbol{d\ell}$$

$$W_1 = -4\times 10^{-3}\int_6^0 xdx$$

$$W_1 = -4\times 10^{-3}\times \left(\frac{x^2}{2}\right)_6^0$$

$$W_1 = 72\, m\,\mathrm{J}$$

Consider Path C_2 :

$$x = 0\,,\quad dx = 0$$

$$z = 0\,,\quad dz = 0$$

$$0 \le y \le 8$$

$$\boldsymbol{d\ell} = dx\boldsymbol{a_x} + dy\boldsymbol{a_y} + dz\boldsymbol{a_z}$$

$$\boldsymbol{d\ell} = dy\boldsymbol{a_y}$$

$$\boldsymbol{E}.\boldsymbol{d\ell} = 4ydy$$

$$W_2 = -Q\int \boldsymbol{E}.\boldsymbol{d\ell}$$

$$W_2 = -8\times 10^{-3}\int_8^0 ydy$$

$$W_2 = -8\times 10^{-3}\times \left(\frac{y^2}{2}\right)_8^0$$

$$W_2 = 256\, m\,\mathrm{J}$$

$$W = W_1 + W_2$$

$$W = 0.328\,J$$

4-3 Voltage or Potential Difference

We want to move the charge of Q coulomb from the initial position B to the final position A against the field of the source as shown in Figure 4-6.

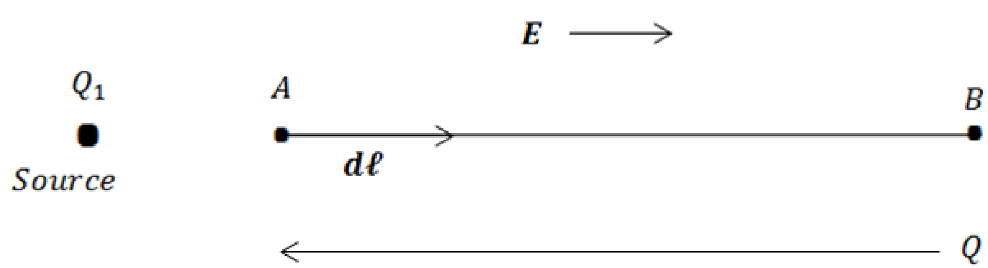


Figure 4-6: Moving a Charge of Q coulomb against the Field

The amount of energy which is required to move the charge of Q coulomb from the initial position B to the final position A against the field is given by

$$W = -Q\int_{B}^{A} \boldsymbol{E}.\boldsymbol{d\ell} \tag{4.19}$$

This energy is electrical energy. The amount of energy which is required to move a unit positive charge from the initial position B to the final position A against the field is known as the voltage or potential difference between points A and B as shown in Figure 4-7.

Thus voltage V_{AB} is calculated as

$$V_{AB} = \frac{W}{Q} = -\int_{B}^{A} \boldsymbol{E}.\boldsymbol{d\ell} \tag{4.20}$$

Figure 4-7: Moving a unit Positive Charge against the Field

Question 4.2:

The two points $Q(6,8,0)$ and $P(0,0,0)$ are located in the electric field intensity $\boldsymbol{E} = 2x\boldsymbol{a}_x + 2y\boldsymbol{a}_y$ V/m, find V_{PQ}.

Solution:

Consider the straight segments from point $Q(6,8,0)$ to $P(0,0,0)$ as shown in Figure 4-8.

Consider Path C_1 :

$y = 8\,, \quad dy = 0$

$z = 0\,, \quad dz = 0$

$0 \leq x \leq 6$

$\boldsymbol{d\ell} = dx\boldsymbol{a}_x + dy\boldsymbol{a}_y + dz\boldsymbol{a}_z$

$\boldsymbol{d\ell} = dx\boldsymbol{a}_x$

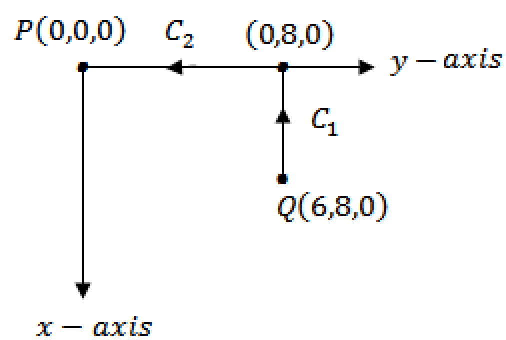


Figure 4-8: Straight Segments

$$\boldsymbol{E}.\boldsymbol{d\ell} = 2xdx$$

Using

$$V_1 = -\int \boldsymbol{E}.\boldsymbol{d\ell}$$

$$V_1 = -2\int_6^0 x dx$$

$$V_1 = -2\left(\frac{x^2}{2}\right)_6^0$$

$$V_1 = 36\ v$$

Consider Path C_2 :

$$x = 0\,,\quad dx = 0$$

$$z = 0\,,\quad dz = 0$$

$$0 \le y \le 8$$

$$\boldsymbol{d\ell} = dx\boldsymbol{a_x} + dy\boldsymbol{a_y} + dz\boldsymbol{a_z}$$

$$\boldsymbol{d\ell} = dy\boldsymbol{a_y}$$

$$\boldsymbol{E}.\boldsymbol{d\ell} = 4ydy$$

$$V_2 = -\int \boldsymbol{E}.\boldsymbol{d\ell}$$

$$V_2 = -2\int_8^0 ydy$$

$$V_2 = -2\left(\frac{y^2}{2}\right)_8^0$$

$$V_2 = 64\ v$$

$$V = V_1 + V_2$$

$$V = 100\ v$$

4-4 Voltage due to a Point Charge

The electric field intensity at any point in the field of the source Q is given by

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon\, r^2}\boldsymbol{a_r} \tag{4.21}$$

Consider two points A and B in the field of the source as shown in Figure 4-9. We know that the amount of energy which is required to move a unit positive charge from the initial position B to the final position A against the field of the source is known as the voltage between points A and B. The voltage V_{AB} is calculated as

$$V_{AB} = -\int_B^A \boldsymbol{E}.\boldsymbol{d\ell} \tag{4.22}$$

Consider $\boldsymbol{d\ell}$ in the direction of intensity

$$\boldsymbol{d\ell} = dr\boldsymbol{a_r}$$

Therefore

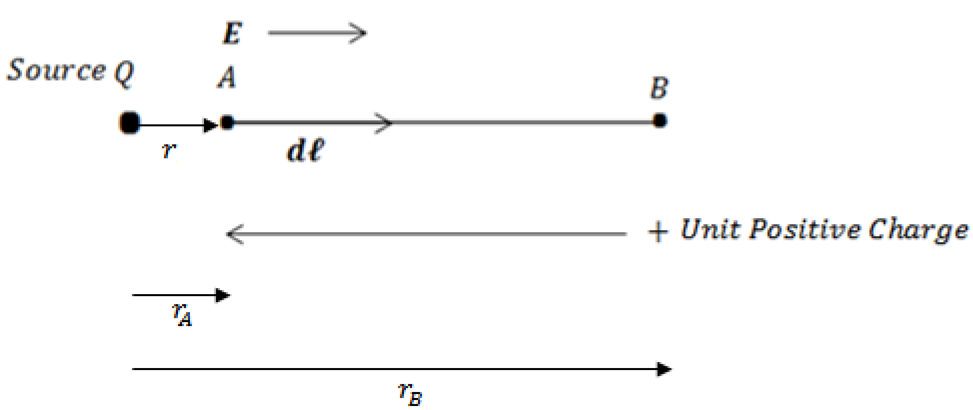


Figure 4-9: Voltage between Points A and B

$$\boldsymbol{E}.\boldsymbol{d\ell} = \frac{Q \times dr}{4\pi\varepsilon\, r^2} \tag{4.23}$$

Putting this value in equation 4.22, we obtain

$$V_{AB} = -\int_{r_B}^{r_A} \frac{Q \times dr}{4\pi\varepsilon\, r^2} \tag{4.24}$$

$$V_{AB} = -\frac{Q}{4\pi\varepsilon}\int_{r_B}^{r_A} \frac{dr}{r^2} \tag{4.25}$$

$$V_{AB} = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r}\right]_{r_B}^{r_A}$$

Thus the voltage between A and B is given by

$$V_{AB} = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]$$

As

$$V_{AB} = V_A - V_B \tag{4.26}$$

Therefore absolute potential at point A

$$V_A = \frac{Q}{4\pi\varepsilon\, r_A} \tag{4.27}$$

Therefore absolute potential at point B

$$V_B = \frac{Q}{4\pi\varepsilon\, r_B} \tag{4.28}$$

The generic form of the potential that is caused by a point charge is given by

$$V = \frac{Q}{4\pi\varepsilon\, r} \tag{4.29}$$

Consider a point charge particle Q as shown in Figure 4-10. Position vector of the source Q is $\boldsymbol{r_1}$ and position vector of P is $\boldsymbol{r}$. The distance vector extending from Q to P is $(\boldsymbol{r} - \boldsymbol{r_1})$. The potential at point P is given by

$$V = \frac{Q}{4\pi\varepsilon\, |\boldsymbol{r} - \boldsymbol{r_1}|} \tag{4.30}$$

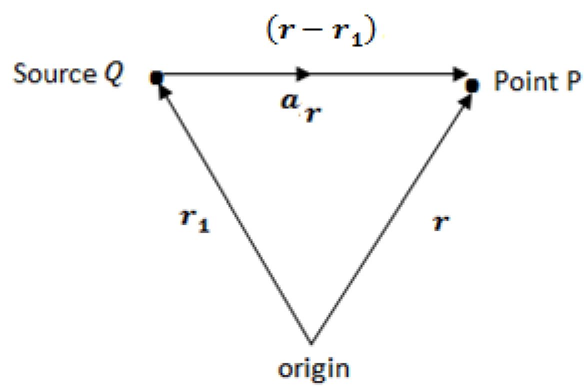


Figure 4-10: Point Charge

Question 4-3:

The two points $B(0,8,6)$ and $A(0,4,3)$ are located in the electric field intensity of a point charge of $Q = 4nC$, if this charge is located at the origin, then (a) find V_{AB} and (b) find V_A if $V = 0$ at ∞.

Solution:

$$r_A = 5m \quad \text{and} \quad r_B = 10m$$

As

$$V_{AB} = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]$$

Therefore

$$V_{AB} = 36\left[\frac{1}{5} - \frac{1}{10}\right]$$

$$V_{AB} = 3.6\ V$$

As

$$V_A = \frac{Q}{4\pi\varepsilon\, r_A}$$

$$V_A = \frac{36}{5} = 7.2\ V$$

4-5 Potential due to n Point Charges

Consider n charge particles as shown in Figure 4-11. Position vector of Q_1 is $\boldsymbol{r_1}$ and position vector of Q_2 is $\boldsymbol{r_2}$ and so on the position vector of Q_n is $\boldsymbol{r_n}$. We want to find out the potential at point P that is caused by all these n number of sources. Position vector of P is $\boldsymbol{r}$.

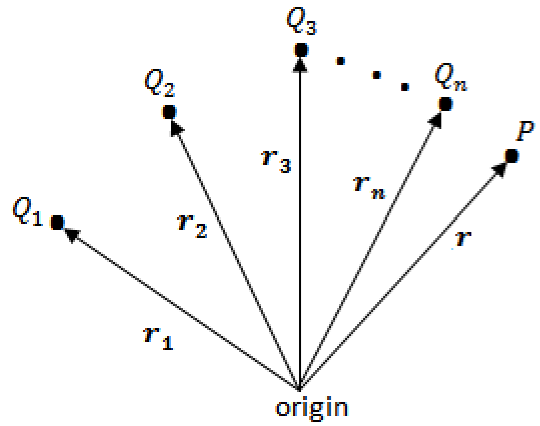


Figure 4-11: n Charge Particles

In order to find out the potential at point P that is caused by all these n number of sources, we apply Superposition Theorem. We assume that there is only one charge that is Q_1 in the vicinity of point P and all the remaining sources do not exist as shown in Figure 4-12. The distance vector extending from Q_1 to P is $\boldsymbol{r} - \boldsymbol{r_1}$.

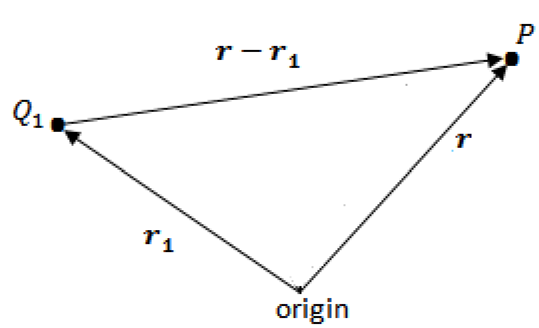


Figure 4-12: Potential due to Q_1

Potential due to Q_1 is given by

$$V_1 = \frac{Q_1}{4\pi\,\varepsilon\,|\boldsymbol{r}-\boldsymbol{r_1}|} \tag{4.31}$$

Now, we assume that there is only one charge that is Q_2 in the vicinity of point P and all the remaining sources do not exist as shown in Figure 4-13. The distance vector extending from Q_2 to P is $\boldsymbol{r}-\boldsymbol{r_2}$.

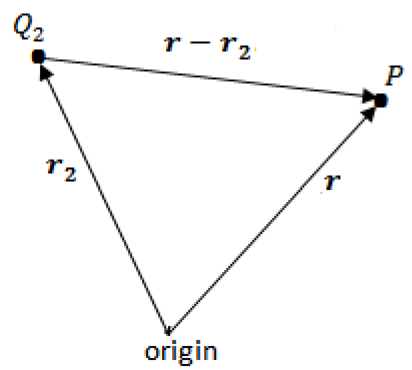


Figure 4-13: Potential due to Q_2

Potential due to Q_2 is given by

$$V_2 = \frac{Q_2}{4\pi\varepsilon\,|\boldsymbol{r}-\boldsymbol{r_2}|} \tag{4.32}$$

Now, let us assume that there is only one charge that is Q_3 in the vicinity of point P and all the remaining sources do not exist as shown in Figure 4-14. The distance vector extending from Q_3 to P is $\boldsymbol{r}-\boldsymbol{r_3}$.

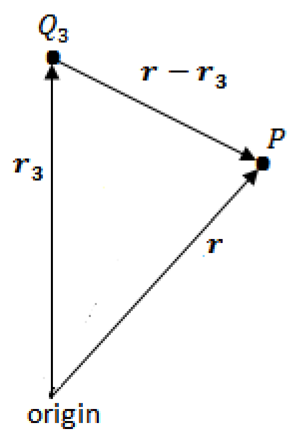


Figure 4-14: Potential due to Q_3

Potential due to Q_3 is given by

$$V_3 = \frac{Q_3}{4\pi\varepsilon\,|\boldsymbol{r}-\boldsymbol{r_3}|} \tag{4.33}$$

Similarly Potential at point P due to Q_n is given by

$$V_n = \frac{Q_n}{4\pi\varepsilon\,|\boldsymbol{r}-\boldsymbol{r_n}|} \tag{4.34}$$

The sum of all these potentials results in the total potential at point P.

$$V = V_1 + V_2 + V_3 + \cdots + V_n$$

$$V = \frac{1}{4\pi\varepsilon}\left[\frac{Q_1}{|\boldsymbol{r}-\boldsymbol{r_1}|} + \frac{Q_2}{|\boldsymbol{r}-\boldsymbol{r_2}|} + \frac{Q_3}{|\boldsymbol{r}-\boldsymbol{r_3}|} + \cdots \frac{Q_n}{|\boldsymbol{r}-\boldsymbol{r_n}|}\right]$$

The last equation in concise form is given by

$$\boldsymbol{D} = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{n}\frac{Q_i}{|\boldsymbol{r}-\boldsymbol{r_i}|}$$

4-6 Potential due to a Line Charge

Assume that charge is uniformly distributed along the length of a line as shown in Figure 4-15. Differential charge on the differential portion of the line is

$$dQ = \rho_L d\ell \tag{4.35}$$

In order to find out the potential at point P that is caused by the line charge, we consider small portion of the source first and determine the potential at P that is caused by the small portion. Position vector of the small portion of the source is $\boldsymbol{r_1}$and position vector of Point P is $\boldsymbol{r}$**.** The distance vector extending from dQ to P is $(\boldsymbol{r}-\boldsymbol{r_1})$**.**
Differential potential at point P that is caused by the differential charge is given by

$$dV = \frac{dQ}{4\pi\varepsilon\,|\boldsymbol{r} - \boldsymbol{r_1}|} \tag{4.36}$$

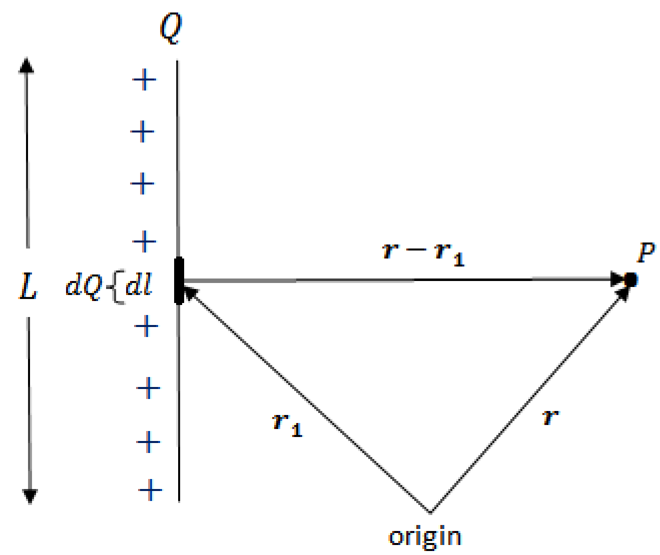


Figure 4-15: Potential due to Line Charge

Putting the value of dQ in equation 36, we obtain

$$dV = \frac{\rho_L d\ell}{4\pi\varepsilon\,|\boldsymbol{r} - \boldsymbol{r_1}|} \tag{4.37}$$

Total potential that is caused by the entire source

$$V = \int \frac{\rho_L d\ell}{4\pi\varepsilon\,|\boldsymbol{r} - \boldsymbol{r_1}|} \tag{4.38}$$

Example 4-4:

The point $P(4, 0, 0)$ is located in the electric field intensity of a line charge of $\rho_L = 20\ nc/m$, if this line charge is located at $y = 0, z = 0$ and $-1 \leq x \leq 1$, then find V_P.

Solution

The potential that is caused by the entire source is given by

$$V_P = \int \frac{\rho_L d\ell}{4\pi\varepsilon\, |\boldsymbol{r} - \boldsymbol{r_1}|}$$

Consider Figure 4-16

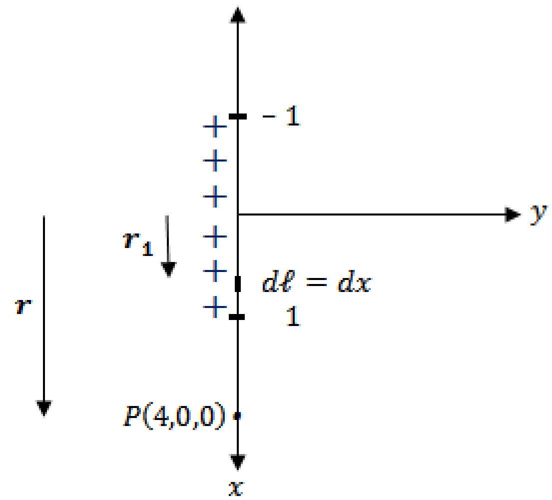


Figure 4-16 for Example 4-4

$$d\ell = dx$$

$$|\boldsymbol{r} - \boldsymbol{r_1}| = 10 - x$$

$$V_P = -20 \times 9 \int_{-1}^{1} \frac{-dx}{10 - x}$$

$$V_P = -180 \ln(10 - x)_{-1}^{1}$$

$$V_P = 36\, V$$

Example 4-5

The point $P(4, 0, 0)$ is located in the electric field intensity of a line charge of

$\rho_L = 4\, nc/m$, if this line charge is located at $y = 0, z = 0$ and $-3 \le z \le 3$, then find V_P.

Solution

The potential that is caused by the entire source is given by

$$V_P = \int \frac{\rho_L d\ell}{4\pi\varepsilon\, |\boldsymbol{r} - \boldsymbol{r_1}|}$$

Consider Figure 4-17

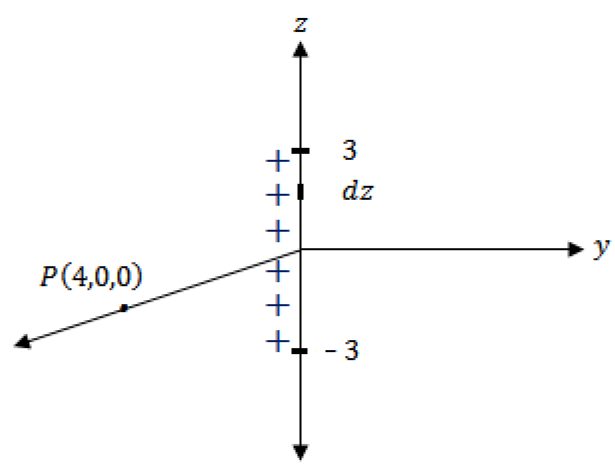


Figure 4-17 for Example 4-5

$$\rho = 4$$

$$d\ell = dz$$

$$|\boldsymbol{r} - \boldsymbol{r_1}| = \sqrt{\rho^2 + z^2}$$

$$V_P = 36 \int_{-3}^{3} \frac{dz}{\sqrt{\rho^2 + z^2}}$$

Let

$$z = \rho \tan\theta$$

$$dz = \rho \sec^2 \theta \, d\theta$$

$$V_P = 36 \int \frac{\rho \sec^2 \theta \, d\theta}{\rho \sec \theta}$$

$$V_P = 36 \int \sec \theta \, d\theta$$

$$V_P = 36 \ln \left(\sec \theta + \tan \theta \right)$$

$$V_P = 36 \ln \left(\frac{z + \sqrt{\rho^2 + z^2}}{\rho} \right)_{-3}^{3}$$

$$V_P = 49.9 \, V$$

4-7 Potential due to Surface Charge

Assume that charge is uniformly distributed along surface of a sheet as shown in Figure 4-18. Total charge on surface S of the sheet is Q coulomb. Charge per unit area is known as surface charge density which is represented by ρ_S. Differential charge on the differential portion of the surface is

$$dQ = \rho_S \, ds \tag{4.39}$$

In order to find out the potential at point P that is caused by the surface charge, we consider small portion of the source first and determine potential at P that is caused by the small portion. Position vector of the small portion of the source is $\boldsymbol{r_1}$ and position vector of Point P is $\boldsymbol{r}$. The distance vector extending from dQ to P is $(\boldsymbol{r} - \boldsymbol{r_1})$. Differential potential at point P that is caused by the differential charge is given by

$$dV = \frac{dQ}{4\pi\varepsilon \, |\boldsymbol{r} - \boldsymbol{r_1}|} \tag{4.40}$$

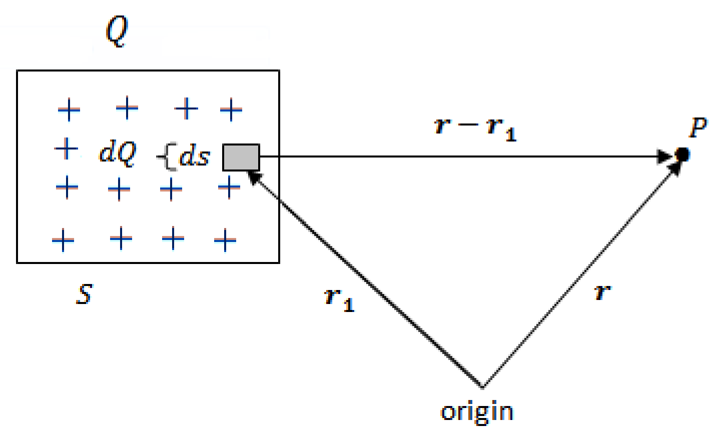


Figure 4-18: Potential due to Surface Charge

Putting the value of dQ in equation 4-40, we obtain

$$dV = \frac{\rho_S \, ds}{4\pi\varepsilon \, |\boldsymbol{r}-\boldsymbol{r_1}|} \tag{4.41}$$

Total potential that is caused by the entire source

$$V = \int \frac{\rho_S \, ds}{4\pi\varepsilon \, |\boldsymbol{r}-\boldsymbol{r_1}|} \tag{4.42}$$

4-8 Potential due to Volume Charge

Assume that charge is uniformly distributed inside a volume V as shown in Figure 4-19. Total charge inside the given volume is Q coulomb. Charge per unit volume is known as volume charge density which is represented by ρ_v. Differential charge in the differential portion of the volume is

$$dQ = \rho_v \, dv \tag{4.43}$$

In order to find out the potential at point P that is caused by the surface charge, we consider small portion of the source first and determine potential at P that is caused by the small portion. Position vector of the small portion of the source is $\boldsymbol{r_1}$ and position vector of Point P is $\boldsymbol{r}$. The distance vector extending from dQ to P is $(\boldsymbol{r}-\boldsymbol{r_1})$.

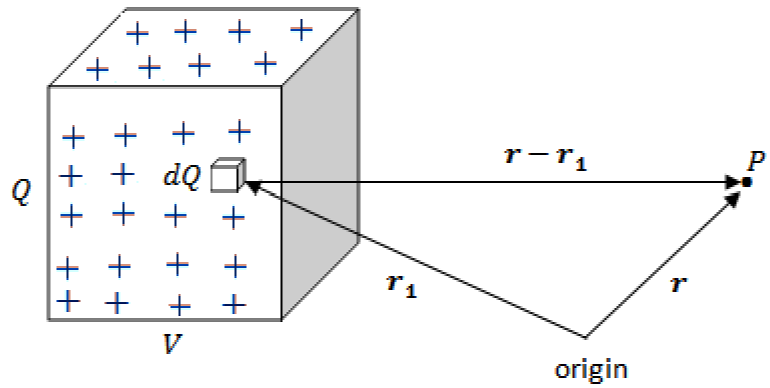


Figure 4-19: Potential due to Volume Charge

Differential potential at point P that is caused by the differential charge is given by

$$dV = \frac{dQ}{4\pi\varepsilon\,|\boldsymbol{r}-\boldsymbol{r_1}|} \qquad (4.44)$$

Putting the value of dQ in equation 4.44, we obtain

$$dV = \frac{\rho_V\, dV}{4\pi\varepsilon\,|\boldsymbol{r}-\boldsymbol{r_1}|} \qquad (4.45)$$

Total potential that is caused by the entire source

$$V = \int \frac{\rho_V\, dV}{4\pi\varepsilon\,|\boldsymbol{r}-\boldsymbol{r_1}|} \qquad (4.46)$$

4-9 Potential Difference due to Infinite Line Charge

Consider an infinite line charge extending from $-\infty$ to ∞ along $z-axis$ as shown in Figure 4-20. Line charge density of the source is ρ_L.The radial distance between the source and point A is ρ. Electric field intensity at point P that is caused by the infinite line charge is given by

$$\boldsymbol{E} = \frac{\rho_L}{2\pi\varepsilon\rho}\boldsymbol{a_\rho} \qquad (4.47)$$

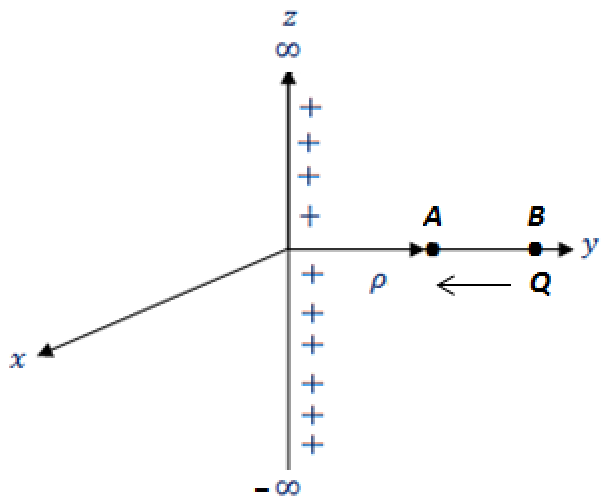


Figure 4-20: Potential due to Infinite Line Charge

The total energy which is required to move the charge of Q coulomb from the initial position B to the final position A against the field is given by

$$W = -Q \int_B^A \boldsymbol{E}.\boldsymbol{d\ell} \tag{4.48}$$

Consider $\boldsymbol{d\ell}$ in the direction of intensity

$$\boldsymbol{d\ell} = d\rho \boldsymbol{a}_\rho$$

Therefore

$$\boldsymbol{E}.\boldsymbol{d\ell} = \frac{\rho_L \times d\rho}{2\pi\varepsilon\rho} \tag{4.49}$$

$$W = -Q \int_B^A \frac{\rho_L \times d\rho}{2\pi\varepsilon\rho} \tag{4.50}$$

$$W = \frac{Q\rho_L}{2\pi\varepsilon}\ln\left(\frac{B}{A}\right) \tag{4.51}$$

Thus the voltage between A and B is given by

$$V_{AB} = \frac{\rho_L}{2\pi\varepsilon}\ln\left(\frac{B}{A}\right) \tag{4.51}$$

Example 4-6

The two points $B(0,8,0)$ and $A(0,4,0)$ are located in the electric field intensity of an infinite line charge of $\rho_L = 5.56 \times 10^{-10}\ C/m$ if this line charge is located along $z-axis$, then find V_{AB}.

Solution

The voltage between A and B is given by

$$V_{AB} = \frac{\rho_L}{2\pi\varepsilon}\ln\left(\frac{B}{A}\right)$$

$$V_{AB} = \frac{5.56 \times 10^{-10}}{5.56 \times 10^{-11}}\ln\left(\frac{8}{4}\right)$$

$$V_{AB} = 6.9\ V$$

4-10 Potential Gradient

Consider the source in the form of point charge as shown in Figure 4-21. The electric field intensity at any point P in the field of the source Q is given by

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon\, r^2}\boldsymbol{a_r} \tag{4.52}$$

The potential that is caused by a point charge is given by

$$V = \frac{Q}{4\pi\varepsilon\, r} \tag{4.53}$$

Figure 4-21: A Point in the Field of Point Charge

Gradient of V in Rectangular Coordinate System is given by

$$\nabla V = \frac{\partial V}{\partial x}\boldsymbol{a_x} + \frac{\partial V}{\partial y}\,\boldsymbol{a_y} + \frac{\partial V}{\partial z}\boldsymbol{a_z} \tag{4.54}$$

Gradient of V in Cylindrical Coordinate System is given by

$$\nabla V = \frac{\partial V}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \emptyset}\,\boldsymbol{a}_\emptyset + \frac{\partial V}{\partial z}\,\boldsymbol{a_z} \tag{4.55}$$

Gradient of V in Spherical Coordinate System is given by

$$\nabla V = \frac{\partial V}{\partial r}\boldsymbol{a_r} + \frac{1}{r sin\theta}\frac{\partial V}{\partial \emptyset}\boldsymbol{a}_\emptyset + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\boldsymbol{a_\theta} \tag{4.56}$$

So we determine the Gradient of V in Spherical Coordinate System

$$\nabla V = -\frac{Q}{4\pi\varepsilon\, r^2}\boldsymbol{a_r} \tag{4.57}$$

Comparing equation 4.57 with equation 4.52, we obtain relationship between electric field intensity and potential

$$\boldsymbol{E} = -\nabla V \tag{4.58}$$

4-11 Potential due to Electric Dipole

An electric dipole is located on the $z - axis$ as shown in Figure 4-22. Separation between the positive charge particle and the negative charge particle of the electric dipole is very small and is represented by d. Center of the dipole is located at the origin of the rectangular coordinate system. We want to find potential and electric field intensity at point P. Distance between the positive charge and point P is r_1, distance between the negative charge and point P is r_2 and distance between the center of the electric dipole and P is r. We assume that the point under observation is located far away from the dipole. In order to compute the potential at point P, we apply Superposition Theorem.

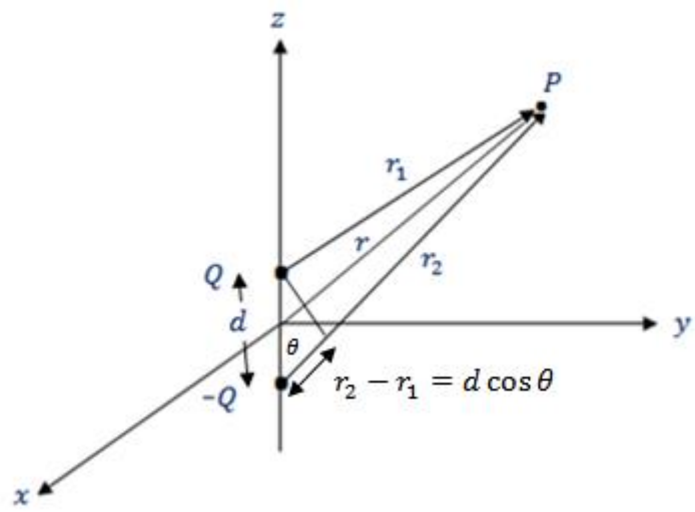


Figure 4-22: Potential due to Electric Dipole

Potential due to positive charge of the dipole is given by

$$V_+ = \frac{Q}{4\pi\varepsilon\, r_1} \tag{4.59}$$

Potential due to negative charge of the dipole is given by

$$V_- = \frac{-Q}{4\pi\varepsilon\, r_2} \tag{4.60}$$

According to Superposition Theorem, the potential at point P is given by

$$V = V_+ + V_- \tag{4.61}$$

$$V = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{4.62}$$

$$V = \frac{Q}{4\pi\varepsilon}\left(\frac{r_2 - r_1}{r_1 r_2}\right) \tag{4.63}$$

As

$$r_2 - r_1 = d\cos\theta$$

Therefore

$$V = \frac{Q}{4\pi\varepsilon}\left(\frac{d\cos\theta}{r_1 r_2}\right) \tag{4.64}$$

As separation between the positive charge particle and the negative charge particle of the electric dipole is very small and the point under observation is located far away from the dipole, so

$$r_2 \approx r_1 \approx r$$

Therefore potential at point P is given by

$$V = \frac{Qd\cos\theta}{4\pi\varepsilon\, r^2} \tag{4.65}$$

Electric field intensity is calculated using the following relationship

$$\boldsymbol{E} = -\nabla V \tag{4.66}$$

$$\boldsymbol{E} = \frac{Qd}{4\pi\varepsilon\, r^3}\left(2\cos\theta\,\boldsymbol{a_r} + \sin\theta\,\boldsymbol{a_\theta}\right) \tag{4.67}$$

Product of Q and d is known as dipole moment. It is a vector quantity that is represented by $\boldsymbol{P}$.

$$\boldsymbol{P} = Q\boldsymbol{d}$$

Where $\boldsymbol{d}$ is the distance vector extending from the negative charge to the positive charge. Consider two unit vectors $\boldsymbol{a_z}$ and $\boldsymbol{a_r}$ as shown in Figure 4-23.

$$\boldsymbol{a_z} \,.\, \boldsymbol{a_r} = \cos\theta$$

Thus

$$\boldsymbol{P}.\boldsymbol{a_r} = Q\boldsymbol{d}.\boldsymbol{a_r} = Qd\boldsymbol{a_z}.\boldsymbol{a_r} = Qd\cos\theta$$

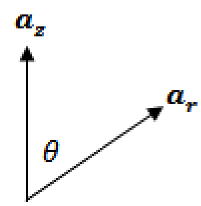


Figure 4-23: A unit Vector along $z - axis$ and a Unit Vector along $\boldsymbol{r}$

Putting this value in equation 4-65, we obtain

$$V = \frac{\boldsymbol{P}.\boldsymbol{a_r}}{4\pi\varepsilon\, r^{\,2}} \qquad (4.68)$$

$$V = \frac{\boldsymbol{P}.\ \boldsymbol{r}}{4\pi\varepsilon\, r^{\,3}} \qquad (4.69)$$

Electric dipole in Figure 4-24 is located away from the origin. Position vector of the center of the dipole is $\boldsymbol{r_1}$. Position vector of point P is $\boldsymbol{r}$. Distance vector extending from center of the dipole to the point under observation is $\boldsymbol{r} - \boldsymbol{r_1}$.

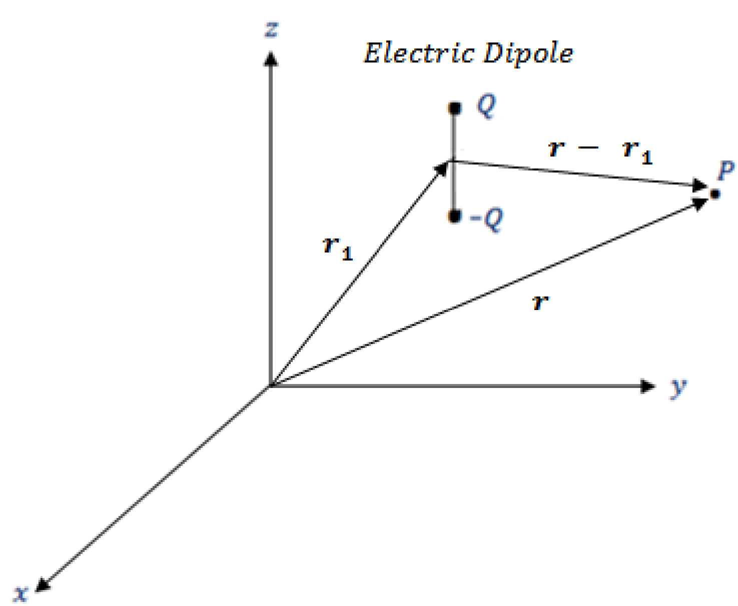


Figure 4-24: Electric Dipole

The potential at point P is given by

$$V = \frac{\boldsymbol{P} \cdot (\boldsymbol{r} - \boldsymbol{r_1})}{4\pi\varepsilon \, |\boldsymbol{r} - \boldsymbol{r_1}|^3} \tag{4.70}$$

Example: 4-7

An electric dipole having $\boldsymbol{P} = 8 \, \boldsymbol{a_z} \, nC - m$ is located at the origin. Find the absolute potential at $B(0, 6, 8)$.

Solution

$$V = \frac{\boldsymbol{P} \cdot (\boldsymbol{r} - \boldsymbol{r_1})}{4\pi\varepsilon_0 \, |\boldsymbol{r} - \boldsymbol{r_1}|^3}$$

$$\boldsymbol{r} - \boldsymbol{r_1} = 6\boldsymbol{a_y} + 8\boldsymbol{a_z}$$

$$|\boldsymbol{r} - \boldsymbol{r_1}| = 10m$$

$$V = \frac{72\boldsymbol{a_z} \cdot (6\boldsymbol{a_y} + 8\boldsymbol{a_z})}{1000}$$

$$V = 576 \, m \, V$$

Example: 4-8

Two point charges of $2nC$ and $-2nC$ are located at ($2mm$, 0, 0) and ($-2mm$, 0, 0) respectively. Find the absolute potential and intensity at $P(2, 45^0, 45^0)$.

Solution

Potential at point P is given by

$$V = \frac{Qd \cos\theta}{4\pi\varepsilon_0 \, r^2}$$

$$V = \frac{8 \times 10^{-12} \times 9 \times 10^9 \times 0.707}{4}$$

$$V = \ 12.73\ mV$$

As

$$\boldsymbol{E} = \frac{Qd}{4\pi\varepsilon_0\, r^{\,3}}\ (2\cos\theta\,\boldsymbol{a_r} + \sin\theta\ \boldsymbol{a_\theta})$$

$$\boldsymbol{E} = 9 \times 10^{-3}(2 \times \cos 45\ \boldsymbol{a_r} +\ sin\ 45\boldsymbol{a_\theta}\,)$$

$$\boldsymbol{E} = 12.72\boldsymbol{a_r} + 6.36\boldsymbol{a_\theta}\ v/m$$

Chapter 5

Magnetic Flux and Magnetic Field Intensity

5-1 Magnetic Flux

Consider an isolated fixed North Pole that can not move from one place to another place as shown in Figure 5-1. There will be magnetic field in the vicinity of this isolated North Pole. If we place a movable isolated north pole in the magnetic field of fixed North Pole it will move along a straight line due the force of repulsion. The path or line followed by an isolated north pole in a magnetic field is known as magnetic flux. It is a scalar quantity and is represented by Ø. The unit of flux is Weber.

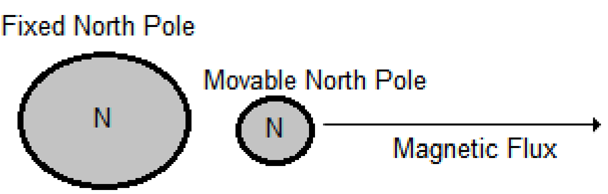


Figure 5-1: Magnetic Flux

We may change the place of the movable isolated North Pole around the fixed one and can trace many more lines. In other words the number of magnetic lines of forces set up in a magnetic circuit is called Magnetic Flux. It is analogous to electric current in an electric circuit. The direction of the arrow gives the direction of the magnetic force, magnetic field intensity and magnetic flux density.

5-2 Magnetic Flux Density

Consider lines of magnetic force Ø passing through a surface S as shown in Figure 5-2. All the lines are normal to the surface. The magnetic flux per unit area defines magnetic flux density and it is represented by B. It is a vector quantity and its unit is $weber/m^2$ or Tesla T. Mathematically

$$B = \frac{\text{Ø}}{S}$$

So

$$\text{Ø} = BS$$

Also

$$\boldsymbol{B} = B\boldsymbol{a_n}$$

$$\boldsymbol{S} = S\boldsymbol{a_n}$$

$$\emptyset = \boldsymbol{B}.\boldsymbol{S}$$

There is another way to compute magnetic flux density. We consider differential magnetic flux $d\emptyset$ passing through a small portion of the given surface that is ds. According to the definition, the magnetic flux per unit area can be calculated as

$$B = \frac{d\emptyset}{ds}$$

Hence the differential magnetic flux passing through the differential area can be computed as under

$$d\emptyset = Bds$$

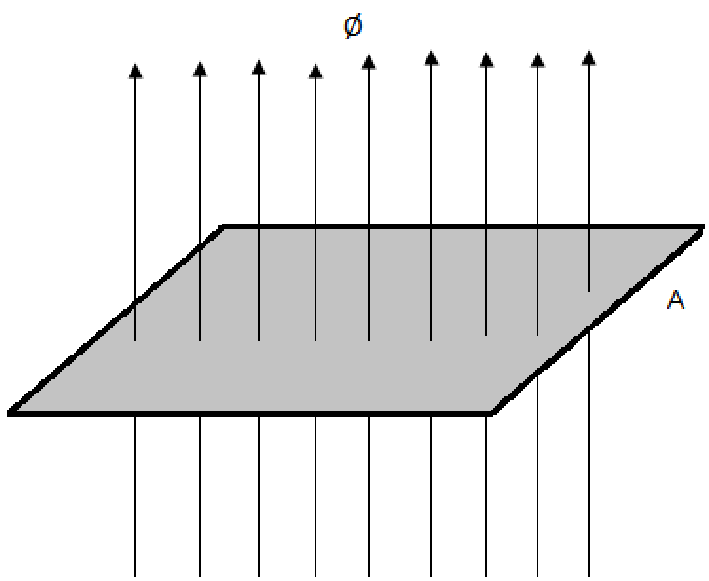


Figure 5-2: Magnetic Flux Density

As

$$\boldsymbol{ds} = ds\boldsymbol{a_n}$$

$$d\emptyset = \boldsymbol{B}.\boldsymbol{ds}$$

$$\emptyset = \int \boldsymbol{B}.\boldsymbol{ds}$$

The flux passing through the closed surface is given by

$$\emptyset = \oint \boldsymbol{B}.\boldsymbol{ds}$$

A source is located inside a sphere as shown in Figure 5-3. As the magnetic flux is continuous, therefore the number of magnetic lines of force entering the closed surface is equal to the number of magnetic lines of force leaving the closed surface. Hence the net magnetic flux in the outward direction from a closed surface is equal to zero.

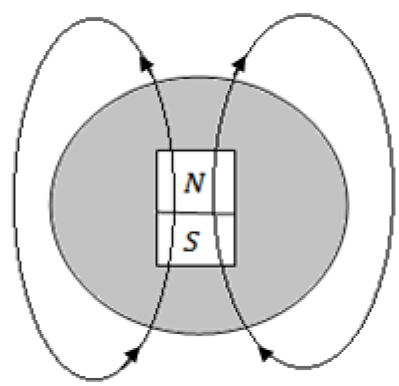


Figure 5-3: Bar Magnet in a Sphere

Mathematically

$$\emptyset = \oint \boldsymbol{B}.\boldsymbol{ds} = 0$$

The Divergence Theorem states that

$$\oint \boldsymbol{B}.\boldsymbol{ds} = \int (\boldsymbol{\nabla}.\boldsymbol{B})\ dv$$

$$\nabla . \boldsymbol{B} = 0 \qquad (5.1)$$

This equation is known as Maxwell's 2nd equation.
Magnetic Flux density is related to magnetic field intensity with the help of following equation.

$$B = \mu H$$

The unit of magnetic field intensity is ampere per meter. It is a vector quantity as well.

Example 5-1:

If $\boldsymbol{B} = 2e^{-z}\,\boldsymbol{a_x}\ T$, then find the flux passing through the following surface in $\boldsymbol{a_x}$ direction.

$$x = 5\,m, \qquad 0 \le y \le 2, \qquad 0 \le z \le 4$$

Solution:

$$\boldsymbol{ds} = dydz\boldsymbol{a_x}$$

$$\boldsymbol{B}.\boldsymbol{ds} = 2e^{-z}\,dzdy$$

$$\emptyset = \int \boldsymbol{B}.\boldsymbol{ds} = 2\int_0^4 e^{-z}\ dz \int_0^2 dy$$

$$\emptyset = 2[-e^{-z}]_0^4 \times [\,y\,]_0^2 \ = 3.93\,wb$$

5-3 Biort -Savart Law

Current in a conductor generates magnetic flux density. Consider a current carrying conductor as shown in Figure 5-4. The magnetic flux density is directly proportional to the current in the conductor, length of the conductor, sine of the angle between the length of the conductor and distance R and is inversely proportional to the square of the distance between the conductor and point under observation.

$$B \propto \frac{IL\sin\theta}{R^2}$$

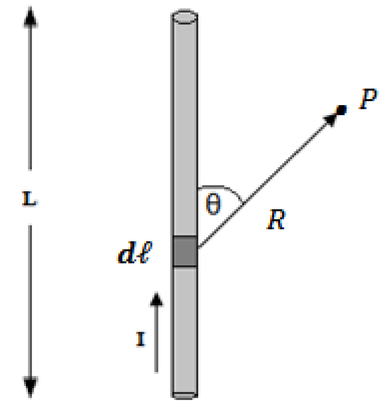


Figure 5-4: Magnetic Field Produced by a Current Carrying Conductor

The differential magnetic flux density that is generated by the current in a small portion of the conductor is given by

$$dB \propto \frac{IdL\sin\theta}{R^2}$$

Magnetic flux density is a vector quantity and its direction is given by

$$\boldsymbol{d\ell} \times \boldsymbol{a_R}$$

Where $\boldsymbol{d\ell}$ is the differential length vector of the conductor in the direction of current and $\boldsymbol{a_R}$ is a unit vector in the direction of R.

$$\boldsymbol{dB} \propto \frac{I\boldsymbol{d\ell} \times \boldsymbol{a_R}}{R^2}$$

$$\boldsymbol{dB} = \frac{\mu}{4\pi}\frac{I\boldsymbol{d\ell} \times \boldsymbol{a_R}}{R^2} \qquad (5.2)$$

The total magnetic flux density at point P is given by

$$\boldsymbol{B} = \frac{\mu}{4\pi}\int \frac{I\boldsymbol{d\ell} \times \boldsymbol{a_R}}{R^2} \qquad (5.3)$$

Consider the current carrying conductor shown in Figure 5-5.

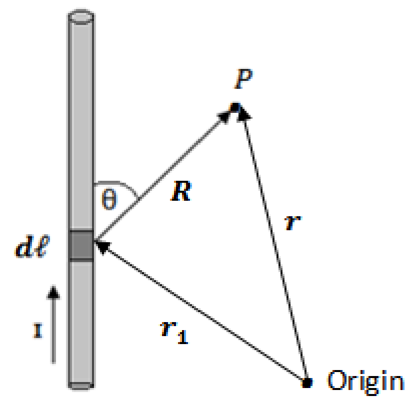


Figure 5-5: Magnetic Field Produced by a Current Carrying Conductor

As

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r_1}$$

$$R = |\boldsymbol{r} - \boldsymbol{r_1}|$$

Therefore

$$\boldsymbol{B} = \frac{\mu}{4\pi} \int \frac{I\boldsymbol{d\ell} \times (\boldsymbol{r} - \boldsymbol{r_1})}{|\boldsymbol{r} - \boldsymbol{r_1}|^3} \tag{5.4}$$

The total magnetic field Intensity at point P is given by

$$\boldsymbol{H} = \frac{1}{4\pi} \int \frac{I\boldsymbol{d\ell} \times (\boldsymbol{r} - \boldsymbol{r_1})}{|\boldsymbol{r} - \boldsymbol{r_1}|^3} \tag{5.5}$$

Example 5.2:

Find the magnetic flux density at $A(0, 4, 0)$, caused by the following source;

$I\boldsymbol{d\ell} = 64 \times 10^{-3}\boldsymbol{a_x}\, Am$ at O(0,0,0)

Solution:

$$\boldsymbol{dB} = \frac{\mu_0}{4\pi} \times \frac{I\boldsymbol{d\ell} \times (\boldsymbol{r} - \boldsymbol{r_1})}{|\boldsymbol{r} - \boldsymbol{r_1}|^3}$$

$$(\boldsymbol{r} - \boldsymbol{r_1}) = 4\,\boldsymbol{a_y}$$

$$\boldsymbol{dB} = 64 \times 10^{-10} \boldsymbol{a_x} \times\ 4\,\boldsymbol{a_y}$$

$$\boldsymbol{dB} = 4 \times 10^{-10} \boldsymbol{a_z}\ T$$

5-4 Intensity due to Infinitely Long Current Carrying Conductor

Consider a current carrying conductor extending from $-\infty$ to ∞ along $z - axis$ as shown in Figure 5-6. We consider a point on the y-axis just for the sake of convenience.

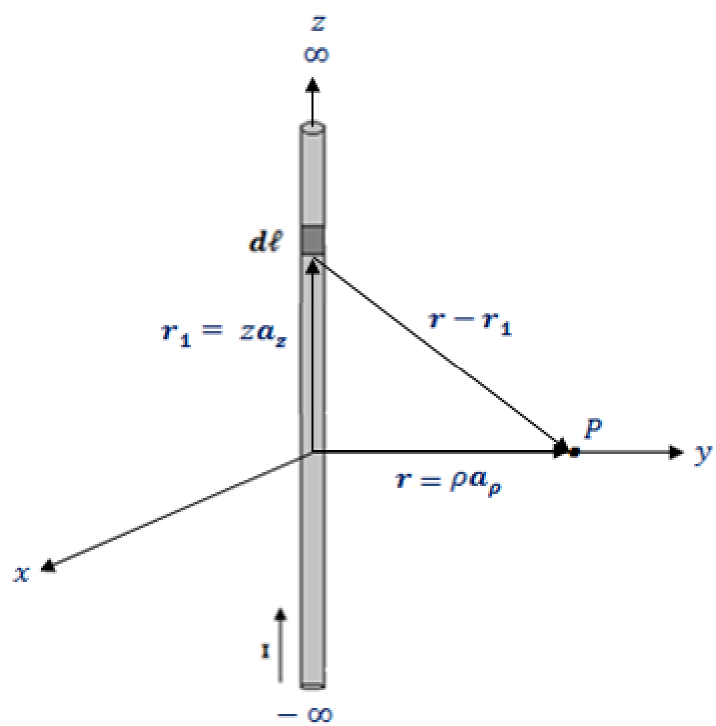


Figure 5-6: Magnetic Field Intensity due to Infinitely Long Current Carrying Conductor

The total magnetic field Intensity at point P is given by

$$\boldsymbol{H} = \frac{1}{4\pi} \int \frac{I\boldsymbol{d\ell} \times (\boldsymbol{r} - \boldsymbol{r_1})}{|\boldsymbol{r} - \boldsymbol{r_1}|^3}$$

As $\qquad \boldsymbol{r} - \boldsymbol{r_1} = \rho \boldsymbol{a_\rho} - z\boldsymbol{a_z}$

And $\quad |\boldsymbol{r}-\boldsymbol{r_1}| = \sqrt{\rho^2 + z^2}$, $\boldsymbol{d\ell} = dz\boldsymbol{a_Z}$

Therefore magnetic field intensity at point P that is caused by the infinite line charge is given by

$$\boldsymbol{H} = \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{Idz\boldsymbol{a_z}\times\ (\rho\boldsymbol{a_\rho} - z\boldsymbol{a_z})}{(\rho^2 + z^2)^{3/2}} \tag{5.6}$$

$$\frac{z}{\rho} = \cot\theta$$

$$z = \rho\cot\theta$$

$$dz = -\rho\,\mathrm{cosec}^2\,\theta\, d\theta$$

When $z = -\infty$, then $\theta = \pi$ and when $z = \infty$, then $\theta = 0$, putting all these values in equation 6, we obtain

$$\boldsymbol{H} = \frac{1}{4\pi}\int_{0}^{\pi}\frac{I\rho\ \mathrm{cosec}^2\,\theta\, d\theta\boldsymbol{a_z}\times(\rho\boldsymbol{a_\rho} - \rho\cot\theta\,\boldsymbol{a_z})}{(\rho^2 + \rho^2\cot^2\theta)^{3/2}} \tag{5.7}$$

$$\boldsymbol{H} = \frac{1}{4\pi}\int_{0}^{\pi}\frac{I\,\rho^2\,\mathrm{cosec}^2\,\theta\, d\theta\,\boldsymbol{a}_\emptyset}{\rho^3(1 + \cot^2\theta\,)^{3/2}} \tag{5.8}$$

$$\boldsymbol{H} = \frac{I}{4\pi\rho}\int_{0}^{\pi}\frac{\mathrm{cosec}^2\,\theta\, d\theta\,\boldsymbol{a}_\emptyset}{(\mathrm{cosec}^2\,)^{3/2}} \tag{5.9}$$

$$\boldsymbol{H} = \frac{I}{4\pi\rho}\int_{0}^{\pi}\frac{d\theta\,\boldsymbol{a}_\emptyset}{cosec\ \theta} \tag{5.10}$$

$$\boldsymbol{H} = \frac{I}{4\pi\rho}\int_{0}^{\pi}\sin\theta\, d\theta\,\boldsymbol{a}_\emptyset$$

$$\boldsymbol{H} = \frac{I}{4\pi\rho} |-\cos\theta|_0^{\pi}\ \boldsymbol{a}_{\emptyset}$$

$$\boldsymbol{H} = \frac{I}{2\pi\rho}\ \boldsymbol{a}_{\emptyset} \qquad (5.11)$$

Example 5-3:

Find the magnetic field intensity at $P(0, 4, 3)$ in rectangular coordinated system caused by a current of $20\pi\ A$ along $z - axis$.

Solution:

$$\boldsymbol{H} = \frac{I}{2\pi\rho}\ \boldsymbol{a}_{\emptyset}$$

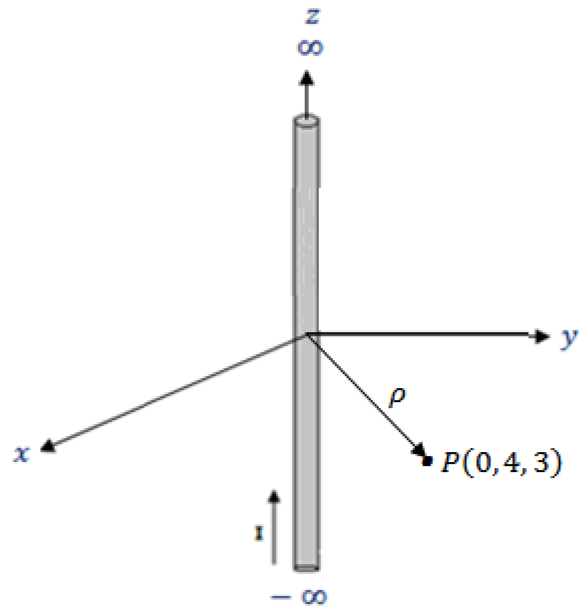
$$\rho = 5\ m$$

$$\boldsymbol{H} = \frac{20\pi}{10\pi}\ \boldsymbol{a}_{\emptyset}$$

$$\boldsymbol{H} = 2\ \boldsymbol{a}_{\emptyset}\ \ A/m$$

$$\emptyset = \tan^{-1}\frac{4}{3}$$

$$\emptyset = 53.1$$



$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \cos 53.1 & -\sin 53.1 & 0 \\ \sin 53.1 & \cos 53.1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\boldsymbol{H} = -1.6\ \boldsymbol{a}_x + 1.6\ \boldsymbol{a}_y\ A/m$$

5-5 Magnetic Field Intensity due to a Small Current Carrying Conductor

Consider a current carrying conductor extending from a to b along $z - axis$ as shown in Figure 5-7. We consider a point on the y-axis just for the sake of convenience. The total magnetic field Intensity at point P is given by

$$\boldsymbol{H} = \frac{1}{4\pi}\int \frac{I\boldsymbol{d\ell} \times (\boldsymbol{r} - \boldsymbol{r_1})}{|\boldsymbol{r} - \boldsymbol{r_1}|^3} \qquad (5.12)$$

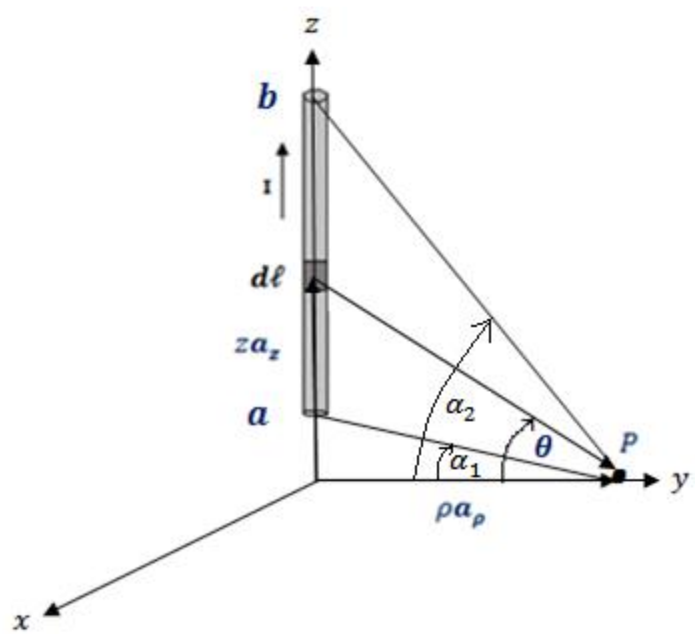


Figure 5-7: Magnetic Field Intensity due to a Small Current Carrying Conductor

As $\quad \boldsymbol{r} - \boldsymbol{r_1} = \rho \boldsymbol{a_\rho} - z\boldsymbol{a_z}$

And $\quad |\,\boldsymbol{r} - \boldsymbol{r_1}| = \sqrt{\rho^2 + z^2} \quad , \boldsymbol{d\ell} = dz\boldsymbol{a_Z}$

Therefore magnetic field intensity at point P that is caused by the small line charge is given by

$$\boldsymbol{H} = \frac{1}{4\pi}\int_{a}^{b} \frac{Idz\boldsymbol{a_z} \times (\rho\boldsymbol{a_\rho} - z\boldsymbol{a_z})}{(\rho^2 + z^2)^{3/2}} \qquad (5.13)$$

$$\frac{z}{\rho} = \tan\theta$$

$$z = \rho\tan\theta$$

$$dz = \rho\sec^2\theta\, d\theta$$

When $z = a$, then $\theta = \alpha_1$ and when $z = b$, then $\theta = \alpha_2$, putting all these values in equation 5.13, we obtain

$$\boldsymbol{H} = \frac{I}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{\rho\ \sec^2\theta\, d\theta\boldsymbol{a_z} \times (\rho\boldsymbol{a_\rho} - \rho\tan\theta\,\boldsymbol{a_z})}{(\rho^2 + \rho^2\tan^2\theta)^{3/2}} \qquad (5.14)$$

$$\boldsymbol{H} = \frac{1}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{I\,\rho^2\,\mathrm{cosec}^2\,\theta\; d\theta\,\boldsymbol{a}_\emptyset}{\rho^3(1 + \tan^2\theta\,)^{3/2}} \qquad (5.15)$$

$$\boldsymbol{H} = \frac{I}{4\pi\rho}\int_{\alpha_1}^{\alpha_2} \frac{\sec^2\theta\; d\theta\,\boldsymbol{a}_\emptyset}{(\sec^2\,)^{3/2}} \qquad (5.16)$$

$$\boldsymbol{H} = \frac{I}{4\pi\rho}\int_{\alpha_1}^{\alpha_2} \frac{d\theta\,\boldsymbol{a}_\emptyset}{sec\,\theta} \qquad (5.17)$$

$$\boldsymbol{H} = \frac{I}{4\pi\rho}\int_{\alpha_1}^{\alpha_2} \cos\theta\, d\theta\,\boldsymbol{a}_\emptyset$$

$$\boldsymbol{H} = \frac{I}{4\pi\rho}\,[\sin\alpha_2 - \sin\alpha_1\,]\,\boldsymbol{a}_\emptyset \qquad (5.18)$$

Example 5-4:

Find the magnetic field intensity at $P(0,6,0)$ in rectangular coordinate system caused by a current of $24\pi\ A$ along $z - axis$. The conductor extends from $z = -8\ \ to\ \ z = 8$.

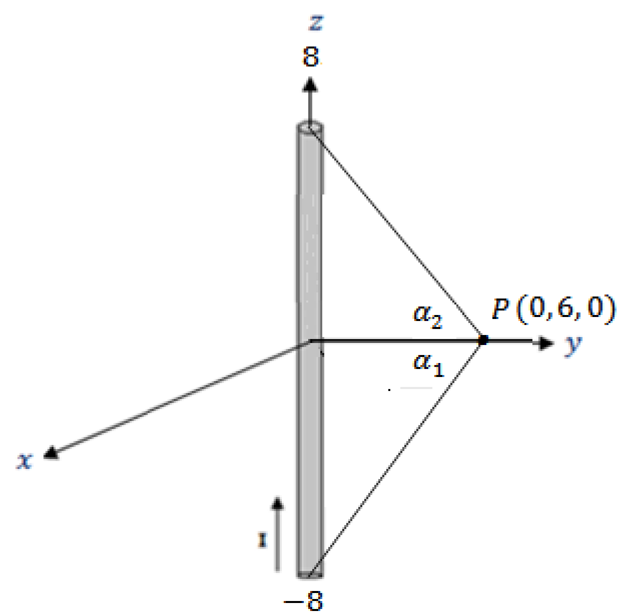
Solution:

$$\boldsymbol{H} = \frac{I}{4\pi\rho}\,[\sin\alpha_2 - \sin\alpha_1\,]\,\boldsymbol{a}_\emptyset$$

$$\rho = 5\ m$$

$$\alpha_1 = -tan^{-1}\frac{8}{6} = \ -53.1^0$$

$$\alpha_2 = tan^{-1}\frac{8}{6} = \ 53.1^0$$



$$\boldsymbol{H} = \frac{24\pi}{24\pi}\,[\sin 53.1 + \sin 53.1\,]\,\boldsymbol{a}_\emptyset$$

$$\boldsymbol{H} = 1.6\ \boldsymbol{a}_\emptyset$$

Using Right hand rule

$$\boldsymbol{H} = -1.6\ \boldsymbol{a}_x\ A/m$$

5-6 Ampere's Circuital Law

Consider an infinitely long current carrying conductor as shown in Figure 5-8. The magnetic field intensity at point P is given by

$$\boldsymbol{H} = \frac{I}{2\pi\rho}\,\boldsymbol{a}_{\emptyset} \tag{5.19}$$

Let us consider a closed circular path of radius ρ around the current carrying conductor. The differential length vector of the closed circular path is given by

$$\boldsymbol{d\ell} = \rho d\emptyset\,\boldsymbol{a}_{\emptyset}$$

The scalar product of the two vectors

$$\boldsymbol{H}.\boldsymbol{d\ell} = \frac{I}{2\pi} d\emptyset$$

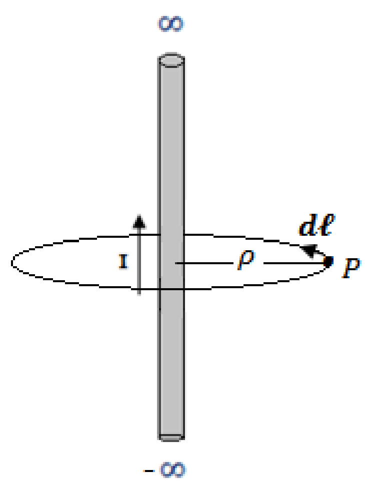


Figure 5-8: Infinitely Long Current Carrying Conductor

The integral of the magnetic field intensity around the closed circular path is given by

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = \frac{I}{2\pi}\int_{0}^{2\pi} d\emptyset$$

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = \frac{I}{2\pi} \times 2\pi$$

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = I_{enclosed} \tag{5.20}$$

We define Ampere's Circuital law in light of the above equation. Ampere's Circuital law states that integral of magnetic field intensity around a closed imaginary path is equal to the current enclosed by the closed path.

Example 5-5:

Find the current in an infinitely long conductor that generates a magnetic field intensity of $2\boldsymbol{a}_\emptyset\ A/m$ along the circumference of a circle around the conductor. Radius of the closed circular path is 5m.

Solution:

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = I_{enclosed}$$

$$\boldsymbol{H} = 2\boldsymbol{a}_\emptyset$$

$$\boldsymbol{d\ell} = \rho d\emptyset\ \boldsymbol{a}_\emptyset$$

$$\boldsymbol{d\ell} = 5d\emptyset\ \boldsymbol{a}_\emptyset$$

$$\boldsymbol{H}.\boldsymbol{d\ell} = 10\ d\emptyset$$

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = 10\int_{\mathbf{0}}^{\mathbf{2\pi}} d\emptyset$$

$$I = 62.8\ A$$

Example 5-6:

Find the integral of the magnetic field intensity around the closed path from $A(2,0,0)$ to $B(2,2,0)$ to $C(4,2,0)$ to $D(4,0,0)$ to $A(2,0,0)$ if the magnetic field intensity is $2\boldsymbol{a}_x + \frac{3}{x}\ \boldsymbol{a}_y\ A/m$.

Solution:

$$\boldsymbol{H} = 2\boldsymbol{a_x} + \frac{3}{x}\,\boldsymbol{a_y}\ A/m$$

Consider path k

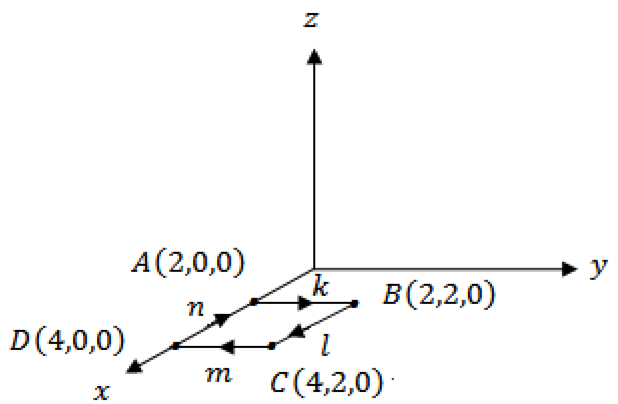
$$x = 2, \qquad so\ \ dx = 0$$

$z = 0, \qquad so\ \ dz = 0$

$0 \leq\ y\ \leq 2$

$\boldsymbol{d\ell} = dy\,\boldsymbol{a_y}$

$\boldsymbol{H}.\boldsymbol{d\ell} = \frac{3}{2}dy$



$$\int_{\boldsymbol{k}} \boldsymbol{H}.\boldsymbol{d\ell} = \frac{3}{2}\int_{0}^{2} dy$$

$$\int_{\boldsymbol{k}} \boldsymbol{H}.\boldsymbol{d\ell} = 3A$$

Consider path l

$$y = 2, \qquad so\ \ dy = 0$$

$$z = 0, \qquad so\ \ dz = 0$$

$$2 \leq\ x\ \leq 4$$

$$\boldsymbol{d\ell} = dx\,\boldsymbol{a_x}$$

$$\boldsymbol{H}.\boldsymbol{d\ell} = 2dx$$

$$\int_{\boldsymbol{l}} \boldsymbol{H}.\boldsymbol{d\ell} = 2\int_{2}^{4} dx$$

$$\int_{\boldsymbol{l}} \boldsymbol{H}.\boldsymbol{d\ell} = 4A$$

Consider path m

$$x = 4, \qquad so\ \ dx = 0$$

$$z = 0, \qquad so\ \ dz = 0$$

$$0 \leq\ y\ \leq 2$$

$$\boldsymbol{d\ell} = dy\,\boldsymbol{a_y}$$

$$\boldsymbol{H}.\boldsymbol{d\ell} = \frac{3}{4}dy$$

$$\int_{\boldsymbol{m}} \boldsymbol{H}.\boldsymbol{d\ell} = \frac{3}{4}\int_{2}^{0} dy$$

$$\int_{\boldsymbol{m}} \boldsymbol{H}.\boldsymbol{d\ell} = -1.5\ A$$

Consider path n

$$y = 0, \qquad so\ \ dy = 0$$

$$z = 0, \qquad so\ \ dz = 0$$

$$2 \leq\ x\ \leq 4$$

$$\boldsymbol{d\ell} = dx\,\boldsymbol{a_x}$$

$$\boldsymbol{H}.\boldsymbol{d\ell} = 2dx$$

$$\int_{n} \boldsymbol{H.d\ell} = 2\int_{4}^{2} dx$$

$$\int_{n} \boldsymbol{H.d\ell} = -4A$$

$$\oint \boldsymbol{H.d\ell} = \int_{k} \boldsymbol{H.d\ell} + \int_{l} \boldsymbol{H.d\ell} + \int_{m} \boldsymbol{H.d\ell} + \int_{n} \boldsymbol{H.d\ell}$$

$$\oint \boldsymbol{H.d\ell} = 1.5\ A$$

5-7 Magnetic Field Intensity due to a Coaxial Cable

Consider a coaxial cable which is extending from $-\infty$ to ∞ along $z - axis$ as shown in Figure 5-9. The radius of the inner conductor is a , the inner radius of the outer conductor is b and the outer radius of the outer conductor is c. There is a dielectric material in between the two conductors of the coaxial cable.

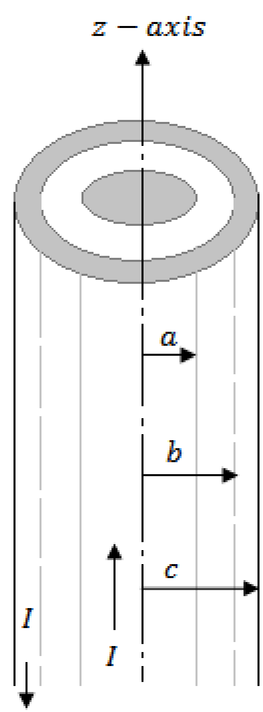


Figure 5-9: Coaxial Cable

The current in the inner conductor of the coaxial cable is I which flows in the direction of positive z-axis, while the same current flows in the opposite direction in the outer conductor. We are going to find out the magnetic field intensity at different points with the help of Ampere's Circuital law. We shall consider a point inside the inner conductor, a point in the dielectric material, a point in the outer conductor and finally a point outside the coaxial cable.

The magnetic field intensity is in the direction of $\boldsymbol{a}_{\emptyset}$.

$$\boldsymbol{H} = H\boldsymbol{a}_{\emptyset}$$

The current density of the inner conductor is given by

$$J_i = \frac{I}{\pi a^2} \tag{5.21}$$

The current density of the outer conductor is given by

$$J_o = \frac{I}{\pi(c^2 - b^2)} \tag{5.22}$$

We consider a point inside the inner conductor as shown in Figure 5-10. The radial distance of the point is ρ. We consider a closed circular path of radius ρ for the application of Ampere's law.

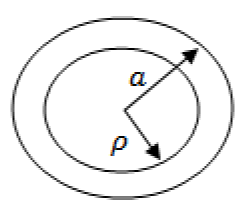


Figure 5-10: Top View of the inner Conductor

Some portion of the inner conductor is located inside the closed circular path. The current enclosed by the closed circular path of radius ρ is given by

$$I_{enclosed} = \frac{I}{\pi a^2} \times \ \pi\rho^2$$

$$I_{enclosed} = \frac{I}{a^2} \times \rho^2 \qquad (5.23)$$

The differential length vector of the closed circular path is given by

$$\boldsymbol{d\ell} = \rho d\emptyset \, \boldsymbol{a}_{\emptyset}$$

The scalar product of the two vectors

$$\boldsymbol{H}.\boldsymbol{d\ell} = H\rho d\emptyset$$

The integral of the magnetic field intensity around the closed circular path is given by

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = H\rho \int_0^{2\pi} d\emptyset$$

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = H\rho \times 2\pi \qquad (5.24)$$

Putting the values in the following equation

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = I_{enclosed}$$

$$H\rho \times 2\pi = \frac{I}{a^2} \times \rho^2$$

$$H = \frac{I\rho}{2\pi a^2}$$

$$\boldsymbol{H} = \frac{I\rho}{2\pi a^2} \boldsymbol{a}_{\emptyset} \qquad (5.25) \ \rho < a$$

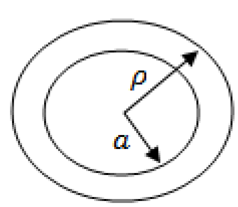


Figure 5-11: Top View When the Point is inside the Dielectric Material

We consider a point inside the dielectric material of the coaxial cable as shown in Figure 5-11. The radial distance of the point is ρ. We consider a closed circular path of radius ρ for the application of Ampere's law.

The entire inner conductor is located inside the closed circular path. The current enclosed by the closed circular path of radius ρ is given by

$$I_{enclosed} = I \qquad (5.26)$$

The differential length vector of the closed circular path is given by

$$\boldsymbol{d\ell} = \rho d\emptyset\, \boldsymbol{a}_{\emptyset}$$

The scalar product of the two vectors

$$\boldsymbol{H}.\boldsymbol{d\ell} = H\rho d\emptyset$$

The integral of the magnetic field intensity around the closed circular path is given by

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = H\rho \int_0^{2\pi} d\emptyset$$

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = H\rho \times 2\pi \qquad (5.27)$$

Putting the values in the following equation

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = I_{enclosed}$$

$$H\rho \times 2\pi = I$$

$$H = \frac{I}{2\pi\rho}$$

$$\boldsymbol{H} = \frac{I}{2\pi\rho}\boldsymbol{a}_{\emptyset} \qquad (5.28)\ a < \rho < b$$

Let us consider a point inside the outer conductor of the coaxial cable. We consider a closed circular path of ρ for the application of Ampere's Circuital Law as shown in Figure 5-12. The entire inner conductor and a portion of the outer conductor are located inside the closed circular path.

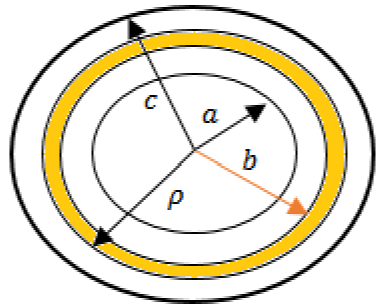


Figure 5-12. Top View When the Point is inside the Outer Conductor

So the current enclosed by the closed path will be equal to the current in the inner conductor plus the current in the shadded region of outer conductor. Both the currents are in opposite directions.

$$I_{enclosed} = I - I_{shad} \tag{5.29}$$

The current in the shadded region of outer conductor is given by

$$I_{shad} = \frac{I}{\pi(c^2 - b^2)} \times \pi(\rho^2 - b^2) \tag{5.30}$$

$$I_{enclosed} = I - \frac{I}{(c^2 - b^2)} \times (\rho^2 - b^2)$$

$$I_{enclosed} = I\frac{(c^2 - \rho^2)}{(c^2 - b^2)} \tag{5.31}$$

As

$$\oint \boldsymbol{H.d\ell} = H\rho 2\pi$$

Putting the values in the following equation

$$\oint \boldsymbol{H.d\ell} = I_{enclosed}$$

$$H\rho 2\pi = I\frac{(c^2 - \rho^2)}{(c^2 - b^2)}$$

$$H = \frac{I}{2\pi\rho}\frac{(c^2 - \rho^2)}{(c^2 - b^2)}$$

$$\boldsymbol{H} = \frac{I}{2\pi\rho}\frac{(c^2 - \rho^2)}{(c^2 - b^2)}\boldsymbol{a}_\emptyset \tag{5.32}$$

$$when\ b < \rho < c$$

Let us consider a point outside the coaxial cable. We consider a closed circular path of ρ for the application of Ampere's Circuital Law as shown in Figure 5-13. The entire coaxial cable is located inside the closed circular path.

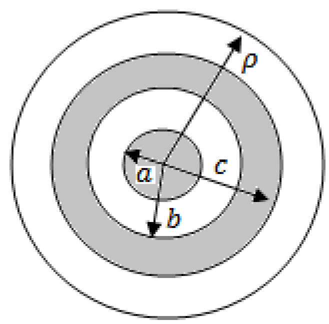


Figure 5-13. Top View When the Point is outside the Cable

So the current enclosed by the closed path will be equal to the current in the inner conductor plus the current in the outer conductor. Both the currents are in opposite directions.

$$I_{enclosed} = I - I = 0 \tag{5.33}$$

As

Putting the values in the following equation

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = I_{enclosed}$$

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = 0$$

$$\boldsymbol{H} = 0$$

Example 5-8:

Find the magnetic field intensity at $A(0, 3mm, 0)$ and $B(0, 5mm, 0)$ caused by an infinite coaxial cable. The inner conductor of the cable carries a current of $4\ mA$ in the $\boldsymbol{a_z}$ direction. The radii of the conductors are;

$$a = 2mm, b = 4mm\ and\ c = 6mm$$

Solution:

$$\boldsymbol{H} = \frac{I}{2\pi\rho}\boldsymbol{a_\emptyset}$$

$$\boldsymbol{H} = \frac{4}{6\pi}\boldsymbol{a_\emptyset}$$

$$\boldsymbol{H} = 0.212\ \boldsymbol{a_\emptyset}\ \ A/m$$

Applying right hand rule, we obtain

$$\boldsymbol{H} = -0.212\ \boldsymbol{a_x}\ \ A/m$$

(b)

$$\boldsymbol{H} = \frac{I}{2\pi\rho}\frac{(c^2 - \rho^2)}{(c^2 - b^2)}\boldsymbol{a_\emptyset}$$

$$\boldsymbol{H} = \frac{4}{10\pi}\frac{(36 - 25)}{(36 - 16)}\boldsymbol{a_\emptyset}$$

$$\boldsymbol{H} = 70\boldsymbol{a_\emptyset}$$

$$\boldsymbol{H} = -70\boldsymbol{a_x}\ \ A/m$$

5-8 Magnetic Field Intensity due to an Infinite Sheet of Current

An infinite current carrying sheet which is located in $\boldsymbol{z} = 0$ plane is shown in Figure 5-14. The current in the sheet flows in the direction of $y - axis$. The current per unit width of the sheet is known as surface current density that is represented by K.

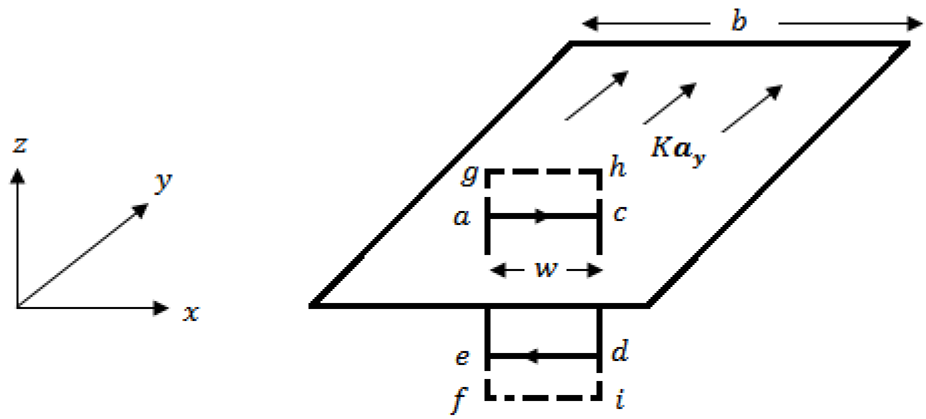


Figure 5-14: Infinite Sheet of Current

$$\boldsymbol{K} = \frac{I}{b}\,\boldsymbol{a_y}$$

We assume that this infinite sheet consists of a large number of infinite current carrying conductors in which the currents flow in the direction of y-axis. Consider two conductors which are located at equal distance from z-axis as shown in Figure 14. Consider a point on the z-axis above the sheet. The magnetic field intensity caused by conductor 1 is $\boldsymbol{H_1}$ and the magnetic field intensity caused by conductor 2 is $\boldsymbol{H_2}$**.** The vector sum of these two fields results in the total magnetic field intensity that is in the direction of x-axis.

$$\boldsymbol{H} = \boldsymbol{H_1} + \boldsymbol{H_2} = H\boldsymbol{a_x}$$

Similarly the magnetic field intensity at a point on the $z - axis$ below the sheet is given by

$$\boldsymbol{H} = -H\boldsymbol{a_x}$$

For the application of Ampere's circuital law, we consider a closed rectangular path a-c-d-e-a as shown in Figure 5-15. The mathematical model of this law is

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = I_{enclosed}$$

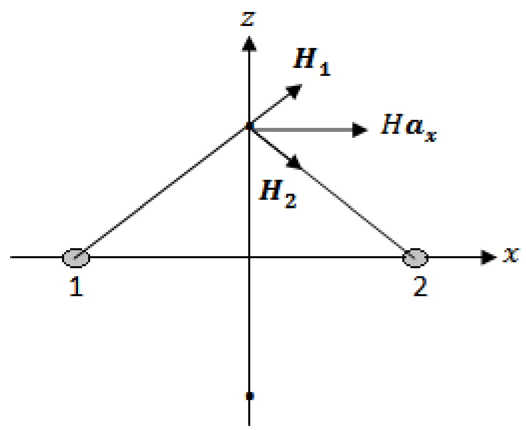


Figure 15: Magnetic Field Intensity above the sheet

Let us evaluate its left hand side,

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = \boldsymbol{H}.\boldsymbol{w} + \boldsymbol{H}.\boldsymbol{w}$$

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = (H\boldsymbol{a}_x\,.w\boldsymbol{a}_x) + (-H\boldsymbol{a}_x\,.-w\boldsymbol{a}_x)$$

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = 2Hw \qquad (5.34)$$

Let us consider another closed rectangular path $f-g-h-i-f$ as shown in Figure 5-14. The mathematical model of Ampere Circuital law is

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = I_{enclosed}$$

Let us evaluate its left hand side,

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = \boldsymbol{H}.\boldsymbol{w} + \boldsymbol{H}.\boldsymbol{w}$$

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = (H\boldsymbol{a}_x\,.w\boldsymbol{a}_x) + (-H\boldsymbol{a}_x\,.-w\boldsymbol{a}_x)$$

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = 2Hw \qquad (5.35)$$

The current enclosed by anyone these two closed paths is given by

$$I_{enclosed} = Kw$$

Therefore

$$2Hw = K\,w$$

$$H = \frac{1}{2}K$$

$$\boldsymbol{H} = \frac{1}{2}K\boldsymbol{a_x} \tag{5.36}$$

This last equation can be written as

$$\boldsymbol{H} = \frac{1}{2}K\boldsymbol{a_y} \ \times \boldsymbol{a_z} \tag{5.37}$$

Or

$$\boldsymbol{H} = \frac{1}{2}\,\boldsymbol{K} \times \boldsymbol{a_z}$$

The same equation in generic form can be written as

$$\boldsymbol{H} = \frac{1}{2}\,\boldsymbol{K} \times \boldsymbol{a_n} \tag{5.38}$$

Where $\boldsymbol{a_n}$ is a unit vector normal to the sheet.

Example 5.9:

Find the magnetic field intensity at $C(4,2,0)$, caused by an infinite sheet of current, located in the $x = 0$ plane. The surface current density of the sheet is $8\boldsymbol{a_z}\ A/m$.

Solution:

$$\boldsymbol{H} = \frac{1}{2}\,\boldsymbol{K} \times \boldsymbol{a_n}$$

$$H = \frac{8}{2}\,\boldsymbol{a_z} \times \boldsymbol{a_x}$$

$$\boldsymbol{H} = 4\boldsymbol{a_y}\ \ A/m$$

5-9 Maxwell's 3rd Equation

Consider a current carrying conductor as shown in Figure 5-16. The current per unit area is known as current density. That is

$$J = \frac{I}{S}$$

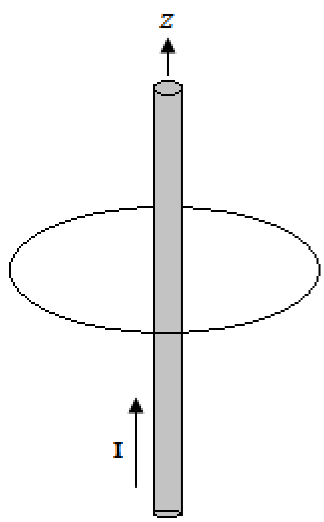


Figure 5-16: Current Carrying Conductor

Where S is the crossectional area of the conductor. As the current in the conductor flows in $\boldsymbol{a_z}$ direction, therefore

$$\boldsymbol{J} = J\boldsymbol{a_z}$$

Consider the crossectional area of the conductor as a vector quantity in the direction of current

$$\boldsymbol{S} = S\boldsymbol{a_z}$$

The current in the conductor is given by

$$I = \boldsymbol{J} \cdot \boldsymbol{S}$$

Let us assume that the current in the small portion of the crossectional area is dI. Then

$$J = \frac{dI}{ds}$$

The small portion of the crossectional area in the direction of current is given by

$$\boldsymbol{dS} = dS\boldsymbol{a_z}$$

Therefore

$$dI = \boldsymbol{J}.\boldsymbol{ds}$$

So, the current in the conductor can be calculated as

$$I = \int \boldsymbol{J}.\boldsymbol{ds} \qquad (5.39)$$

We apply Ampere's circuital law around the closed circular path, the mathematical model of Ampere Circuital law is

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = I_{enclosed}$$

Therefore

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = \int \boldsymbol{J}.\boldsymbol{ds} \qquad (5.40)$$

We apply Stokes theorem on the left hand side

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = \int (\boldsymbol{\nabla} \times \boldsymbol{H}).\boldsymbol{ds}$$

$$\int (\boldsymbol{\nabla} \times \boldsymbol{H}).\boldsymbol{ds} = \int \boldsymbol{J}.\boldsymbol{ds}$$

$$(\boldsymbol{\nabla} \times \boldsymbol{H}) = \boldsymbol{J} \qquad (5.41)$$

This equation is known as Maxwell's 3rd equation. The curl of $\boldsymbol{H}$ in rectangular coordinate system is calculated as

$$\nabla \times \boldsymbol{H} = \begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

The curl of $\boldsymbol{H}$ in cylindrical coordinate system is calculated as

$$\nabla \times \boldsymbol{H} = \frac{1}{\rho} \begin{vmatrix} \boldsymbol{a}_\rho & \rho \boldsymbol{a}_\emptyset & \boldsymbol{a_z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \emptyset} & \frac{\partial}{\partial z} \\ H_\rho & \rho H_\emptyset & H_z \end{vmatrix}$$

The curl of $\boldsymbol{H}$ in spherical coordinate system is calculated as

$$\nabla \times \boldsymbol{H} = \frac{1}{r^2 sin\theta} \begin{vmatrix} \boldsymbol{a}_r & r\boldsymbol{a_\theta} & rsin\theta \boldsymbol{a}_\emptyset \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \emptyset} \\ H_r & rH_\theta & rsin\theta H_\emptyset \end{vmatrix}$$

Example 5.8:

Find $\nabla \times \boldsymbol{H}$ if $\boldsymbol{H} = 2e^{2z}\boldsymbol{a_x}\ \ A/m$

Solution:

$$\nabla \times \boldsymbol{H} = \begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$\nabla \times \boldsymbol{H} = \begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2e^{2z} & 0 & 0 \end{vmatrix}$$

$$\nabla \times \boldsymbol{H} = 4e^{2z}\,\boldsymbol{a_y}\ \ A/m^2$$

Chapter 6

Force and Torque

6-1 Force on a Current Carrying Conductor

When a current carrying conductor is placed in magnetic field, it experiences a force. Force is a vector quantity having magnitude as well as direction. Vector quantities are represented by bold letters. Consider a current carrying conductor which is located in magnetic field as shown in Figure 6-1. This conductor will experience a force which is directly proportional to strength of the magnetic field, length of conductor, current in the conductor and sine of the angle between length and magnetic field.

$$F = ILB \sin\theta \tag{6.1}$$

The above equation means that if current in the conductor is parallel to the magnetic flux density, then no force will be exerted on the conductor. So, proper orientation of the magnetic field plays an important role. As force is a vector quantity, in order to find out the direction of this force, length of the conductor is considered as a vector quantity in the direction of the current.

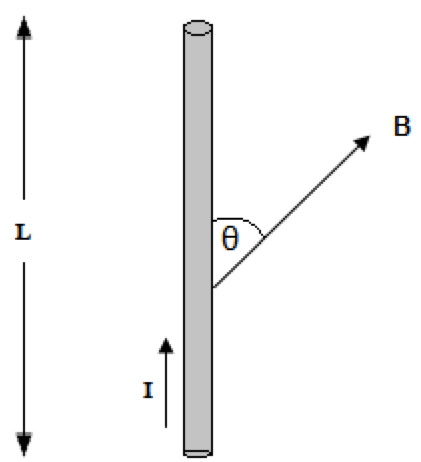


Figure 6-1: Current Carrying Conductor in a Magnetic Field

The direction of the cross product $\boldsymbol{L} \times \boldsymbol{B}$ is the direction of the force on the current carrying conductor. So the direction as well as magnitude of this force can be found with the following equation.

$$\boldsymbol{F} = I\boldsymbol{L} \times \boldsymbol{B} \tag{6.2}$$

Another way to find this force is to consider a very small portion of the current carrying conductor, represented by $\boldsymbol{d\ell}$ as shown in Figure 6-2.

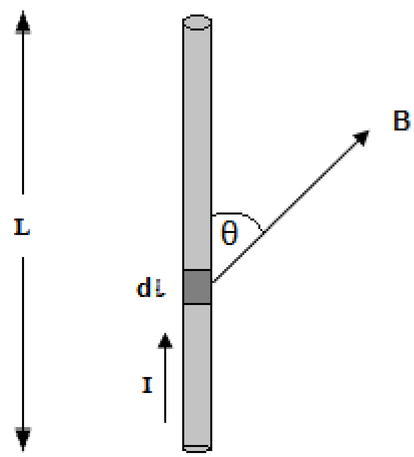


Figure 6-2: Differential Current Element in a Magnetic Field

$\boldsymbol{d\ell}$ is a vector quantity and this differential vector is always in direction of the current. The differential force on the differential portion of the current carrying conductor is calculated with the help of equation 6.3.

$$\boldsymbol{dF} = I\boldsymbol{d\ell} \times \boldsymbol{B} \tag{6.3}$$

If we want to compute the total force on the current carrying conductor, we need to integrate both sides of equation 6.3.

$$\boldsymbol{F} = \int I\boldsymbol{d\ell} \times \boldsymbol{B} \tag{6.4}$$

Example 6.1:

A differential current element having length of $2 \times 10^{-4}\ m$ carries a current of $8A$ in the $\boldsymbol{a_z}$ direction. How much force is experienced by the conductor if it is placed in a magnetic field of $2 \times 10^{-5}\boldsymbol{a_y}\ T$.

Solution:

$$dF = Id\ell \times B$$

$$dF = 16 \times 10^{-4} a_z \times 2 \times 10^{-5} a_y$$

$$dF = -32 a_x \; nN$$

6-2 Force on a Moving Charge

A current carrying conductor is placed in a magnetic field as shown in Figure 6-3. The force which is experienced by this conductor is given by

$$F = \int I d\ell \times B \tag{6.5}$$

Current in this conductor is due to motion of free charge as the rate of motion of charge defines current. The total free charge which is in motion in the above mentioned conductor is Q coulomb.

$$I = \frac{dQ}{dt}$$

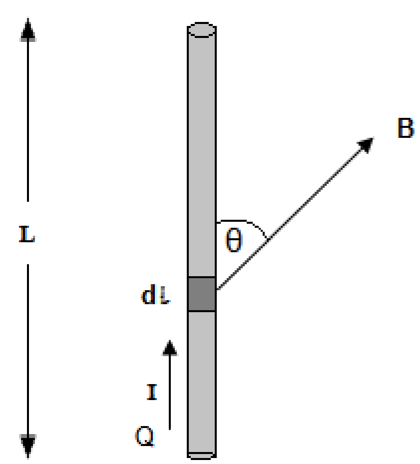


Figure 6-3: Current Carrying Conductor in a Magnetic Field

Multiplying both sides of the above equation by $d\ell$, we obtain the following equation

$$I\boldsymbol{d\ell} = dQ\frac{\boldsymbol{d\ell}}{dt}$$

We assume that the free charge inside the conductor travels distance $\boldsymbol{d\ell}$ in time dt, then $\frac{\boldsymbol{d\ell}}{dt}$ represents velocity of the free charge inside the conductor. Velocity of the free charge is represented by $\boldsymbol{V}$. So

$$I\boldsymbol{d\ell} = dQ\,\boldsymbol{V}$$

In light of the above equation, the force on the current carrying conductor can be found as

$$\boldsymbol{F} = \int dQ\ (\boldsymbol{V} \times \boldsymbol{B}) \tag{6.6}$$

Let us assume that the free charge moves with a uniform velocity $\boldsymbol{V}$ in a uniform magnetic field $\boldsymbol{B}$, then

$$\int dQ = Q$$

So force on the charge that moves in a magnetic field $\boldsymbol{B}$ is given by

$$\boldsymbol{F} = Q\ (\boldsymbol{V} \times \boldsymbol{B}) \tag{6.7}$$

Now, let us assume that charge Q moves in a magnetic as well as electric field as shown in Figure 6-4.

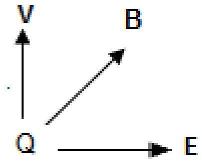


Figure 6-4: Moving Charge in a Magnetic as well as Electric Field

There are two sources that exert force on the moving charge, electric and magnetic field. In order to find the total force on the moving charge we apply Superposition theorem. The Force on the moving charge in the absence of electric field is given by

$$\boldsymbol{F_1} = Q\ (\boldsymbol{V} \times \boldsymbol{B}) \tag{6.8}$$

Force on the moving charge in the absence of magnetic field is given by

$$\boldsymbol{F_2} = Q\boldsymbol{E} \tag{6.9}$$

Vector sum of these two forces results in the total force on the moving charge

$$\boldsymbol{F} = \boldsymbol{F_1} + \boldsymbol{F_2}$$

$$\boldsymbol{F} = Q\ [(\boldsymbol{V} \times \boldsymbol{B}) + \boldsymbol{E}] \tag{6.10}$$

This last equation is known as Lorentz Force Equation.

Example 6-2:

A charge of $2 \times 10^{-3}\ C$ is moving with a uniform velocity of $3\boldsymbol{a_z}\ m/sec$ in a magnetic flux density of $(2\boldsymbol{a_x} - 3\boldsymbol{a_y})\ T$, Find the magnetic force experienced by the charge. (b) If we consider an electric field intensity of $(-2\boldsymbol{a_x} + 2\boldsymbol{a_y})\ \ v/m$, then find the magnitude of the total force on the charge.

Solution:

$$\boldsymbol{F_m} = Q\ (\boldsymbol{V} \times \boldsymbol{B})$$

$$\boldsymbol{F_m} = 6 \times 10^{-3}\boldsymbol{a_z} \times (2\boldsymbol{a_x} - 3\boldsymbol{a_y})$$

$$\boldsymbol{F_m} = (18\boldsymbol{a_x} + 12\boldsymbol{a_y})\ \ mN$$

(b)

We find the electric force on the charge in the absence of magnetic field.

$$\boldsymbol{F_E} = Q\ \boldsymbol{E}$$

$$\boldsymbol{F_E} = (-4\boldsymbol{a_x} + 4\boldsymbol{a_y})\ \ mN$$

According to Superposition Theorem

$$\boldsymbol{F_T} = \ \boldsymbol{F_m} + \boldsymbol{F_E}$$

$$\boldsymbol{F_T} = \left(14\boldsymbol{a_x} + 16\boldsymbol{a_y}\right)\ mN$$

$$\boldsymbol{F_T} = 21.26\, mN$$

6-3 Force between two Current Carrying Conductors

Two current carrying conductors are placed in the magnetic fields of each other as shown as in Figure 6-5. The current I_1 in the first conductor will produce a magnetic field $\boldsymbol{B_1}$ in accordance with Biot-Savart Law and it will exert a force $\boldsymbol{F_2}$ on the second current carrying conductor. Similarly the current I_2 in the second conductor will produce a magnetic field $\boldsymbol{B_2}$ in accordance with Biot-Savart Law and it will exert a force $\boldsymbol{F_1}$ on the first current carrying conductor. The ongoing discussion implies that when two current carrying conductors are placed close to each other then there is either a force of attraction or a force of repulsion between them. The nature of the force depends upon the directions of the two currents which will be explored in the upcoming discussion.

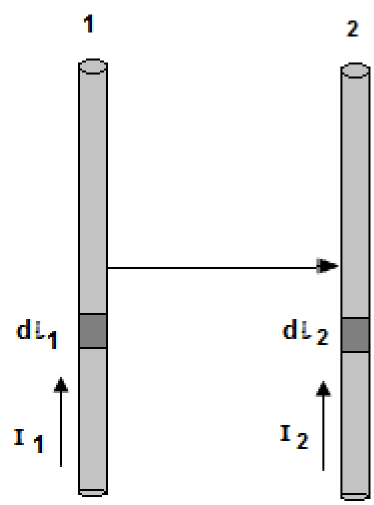


Figure 6-5: Two Current Carrying Conductors

Force experienced by the differential portion of the first conductor due to the magnetic field of the differential portion of second conductor is computed using the following equation.

$$d\,(\boldsymbol{dF_1}) = I_1 \boldsymbol{d\ell_1} \times \boldsymbol{dB_2}$$

Where

$$\boldsymbol{dB_2} = \frac{\mu}{4\pi} \frac{I_2 \boldsymbol{d\ell_2} \times (\boldsymbol{r_1} - \boldsymbol{r_2})}{|\boldsymbol{r_1} - \boldsymbol{r_2}|^3}$$

Total force experienced by the first conductor due to the magnetic field of the second conductor is computed using the following equation.

$$\boldsymbol{F_1} = \int I_1 \boldsymbol{d\ell_1} \times \boldsymbol{B_2} \tag{6.11}$$

Where

$$\boldsymbol{B_2} = \frac{\mu}{4\pi} \int \frac{I_2 \boldsymbol{d\ell_2} \times (\boldsymbol{r_1} - \boldsymbol{r_2})}{|\boldsymbol{r_1} - \boldsymbol{r_2}|^3}$$

Where $\boldsymbol{d\ell_1}$ is the differential length vector of the first current carrying conductor in direction of the current I_1. Experimentally the force of repulsion between two current carrying conductors is found with an apparatus known as Current Balance.

Force experienced by the differential portion of the second conductor due to the magnetic field of the differential portion of first conductor is computed using the following equation.

$$d\,(\boldsymbol{dF_2}) = I_2 \boldsymbol{d\ell_2} \times \boldsymbol{dB_1}$$

Where

$$\boldsymbol{dB_1} = \frac{\mu}{4\pi} \frac{I_1 \boldsymbol{d\ell_1} \times (\boldsymbol{r_2} - \boldsymbol{r_1})}{|\boldsymbol{r_2} - \boldsymbol{r_1}|^3}$$

Total force experienced by the second conductor due to the magnetic field of the first conductor is computed using the following equation.

$$\boldsymbol{F_2} = \int I_2 \boldsymbol{d\ell_2} \times \boldsymbol{B_1} \tag{6.12}$$

Where

$$\boldsymbol{B_1} = \frac{\mu}{4\pi} \int \frac{I_1 \boldsymbol{d\ell_1} \times (\boldsymbol{r_2} - \boldsymbol{r_1})}{|\boldsymbol{r_2} - \boldsymbol{r_1}|^3}$$

Where $\boldsymbol{d\ell_2}$ is the differential length vector of the second current carrying conductor in direction of the current I_2. This fact should be kept in mind that bold letters in these equations denote vector quantities. This is law of nature that things tend to move from a place of higher potential to a place of lower potential. If we apply right hand rule on the two current carrying conductors of Figure 6-6, the magnetic flux cancel the effect of each other in the space between these two conductors.

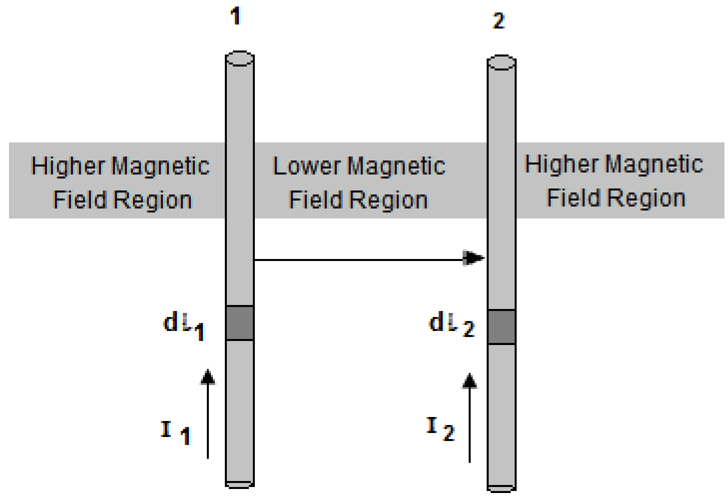


Figure 6-6: Force of Attraction between two Current Carrying Conductors

Obviously these two conductors will tend to move from a place of higher magnetic field to a place of lower magnetic field and there will be a force of attraction between them. So it is concluded that if the currents in these conductors are in same direction, then there will be a force of attraction between the conductors.

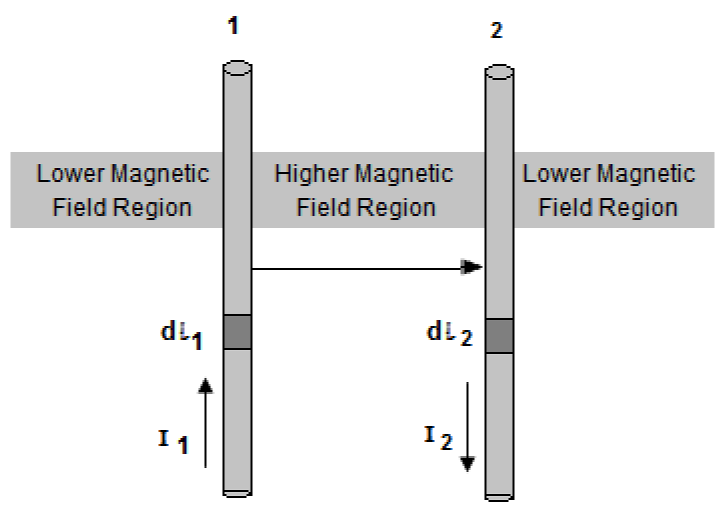


Figure 6-7: Force of Repulsion between two Current Carrying Conductors

If we apply right hand rule on the two current carrying conductors of Figure 6-7, the magnetic flux reinforce the effect of each other in the space between these two conductors. Obviously these two conductors will tend to move from a place of higher magnetic field to a place of lower magnetic field and there will be a force of repulsion between them. So it is concluded that if the currents in these conductors are in opposite direction, then there will be a force of repulsion between the conductors.

Example 6-3:

Two differential current elements;

$I_1 \boldsymbol{d\ell_1} = 8 \times 10^{-4}\, \boldsymbol{a_z}\, Am$ and $I_2 \boldsymbol{d\ell_2} = -8 \times 10^{-4}\, \boldsymbol{a_z}\ \ Am$

are located at $(0, 0, 0)$ and $(0, 2, 0)$ respectively. Calculate $d\ (\boldsymbol{dF_1})$ and $d\ (\boldsymbol{dF_2})$.

Solution:

$$d\ (\boldsymbol{dF_1}) = I_1 \boldsymbol{d\ell_1} \times \boldsymbol{dB_2}$$

Where

$$\boldsymbol{dB_2} = \frac{\mu_0}{4\pi} \frac{I_2 \boldsymbol{d\ell_2} \times (\boldsymbol{r_1} - \boldsymbol{r_2})}{|\boldsymbol{r_1} - \boldsymbol{r_2}|^3}$$

$$\boldsymbol{dB_2} = \frac{-8 \times 10^{-11}\, \boldsymbol{a_z} \times \left(-2\boldsymbol{a_y}\right)}{8}$$

$$\boldsymbol{dB_2} = -2 \times 10^{-11}\, \boldsymbol{a_x} \quad T$$

$$d\ (\boldsymbol{dF_1}) = 8 \times 10^{-4}\, \boldsymbol{a_z} \times -2 \times 10^{-11}\, \boldsymbol{a_x}$$

$$d\ (\boldsymbol{dF_1}) = -16 \times 10^{-15}\, \boldsymbol{a_y} \ \ N$$

(b)

$$d\ (\boldsymbol{dF_2}) = I_2 \boldsymbol{d\ell_2} \times \boldsymbol{dB_1}$$

Where

$$\boldsymbol{dB_1} = \frac{\mu_0}{4\pi}\frac{I_1\boldsymbol{d\ell_1} \times (\boldsymbol{r_2} - \boldsymbol{r_1})}{|\boldsymbol{r_2} - \boldsymbol{r_1}|^3}$$

$$\boldsymbol{dB_1} = \frac{8 \times 10^{-11}\,\boldsymbol{a_z} \times (2\boldsymbol{a_y})}{8}$$

$$\boldsymbol{dB_1} = -2 \times 10^{-11}\,\boldsymbol{a_x} \quad T$$

$$d\,(\boldsymbol{dF_2}) = -8 \times 10^{-4}\,\boldsymbol{a_z} \times -2 \times 10^{-11}\,\boldsymbol{a_x}$$

$$d\,(\boldsymbol{dF_2}) = 16 \times 10^{-15}\,\boldsymbol{a_y} \quad N$$

Where

$$\boldsymbol{dB_2} = \frac{\mu_0}{4\pi}\frac{I_2\boldsymbol{d\ell_2} \times (\boldsymbol{r_1} - \boldsymbol{r_2})}{|\boldsymbol{r_1} - \boldsymbol{r_2}|^3}$$

$$\boldsymbol{dB_2} = \frac{-8 \times 10^{-11}\,\boldsymbol{a_z} \times (-2\boldsymbol{a_y})}{8}$$

$$\boldsymbol{dB_2} = -2 \times 10^{-11}\,\boldsymbol{a_x} \quad T$$

$$d\,(\boldsymbol{dF_1}) = 8 \times 10^{-4}\,\boldsymbol{a_z} \times -2 \times 10^{-11}\,\boldsymbol{a_x}$$

$$d\,(\boldsymbol{dF_1}) = -16 \times 10^{-15}\,\boldsymbol{a_y} \quad N$$

There is a force of repulsion between the two conductors.

Example 6-4:

Two differential current elements;

$I_1 \boldsymbol{d\ell_1} = 2 \times 10^{-3}\, \boldsymbol{a_z}\, Am$ and $I_2 \boldsymbol{d\ell_2} = 4 \times 10^{-3}\, \boldsymbol{a_z} \quad Am$

are located at $(2, 2, 0)$ and $(3,\ 4,\ 0)$ respectively. Calculate $d\,(\boldsymbol{dF_1})$ and $d\,(\boldsymbol{dF_2})$.

Solution:

$$d\,(\boldsymbol{dF_1}) = I_1 \boldsymbol{d\ell_1} \times \boldsymbol{dB_2}$$

Where

$$\boldsymbol{dB_2} = \frac{\mu_0}{4\pi} \frac{I_2 \boldsymbol{d\ell_2} \times (\boldsymbol{r_1} - \boldsymbol{r_2})}{|\boldsymbol{r_1} - \boldsymbol{r_2}|^3}$$

$$\boldsymbol{dB_2} = \frac{4 \times 10^{-10}\, \boldsymbol{a_z} \times \left(-\boldsymbol{a_x} - 2\boldsymbol{a_y}\right)}{(2.24)^3}$$

$$\boldsymbol{dB_2} = 0.356 \times 10^{-10}\, \boldsymbol{a_z} \times \left(-\boldsymbol{a_x} - 2\boldsymbol{a_y}\right)$$

$$\boldsymbol{dB_2} = \left(0.71\boldsymbol{a_x} - 0.356\boldsymbol{a_y}\right) \times 10^{-10} \quad T$$

$$d\,(\boldsymbol{dF_1}) = 2 \times 10^{-3}\, \boldsymbol{a_z} \times \left(0.71\boldsymbol{a_x} - 0.356\boldsymbol{a_y}\right) \times 10^{-10}$$

$$d\,(\boldsymbol{dF_1}) = \left(0.71\boldsymbol{a_x} + 1.42\boldsymbol{a_y}\right) \times 10^{-13}\, N$$

(b)

Solution:

$$d\,(\boldsymbol{dF_2}) = I_2 \boldsymbol{d\ell_2} \times \boldsymbol{dB_1}$$

Where

$$\boldsymbol{dB_1} = \frac{\mu_0}{4\pi} \frac{I_1 \boldsymbol{d\ell_1} \times (\boldsymbol{r_2} - \boldsymbol{r_1})}{|\boldsymbol{r_2} - \boldsymbol{r_1}|^3}$$

$$\boldsymbol{dB_1} = \frac{2 \times 10^{-10}\, \boldsymbol{a_z} \times \left(\boldsymbol{a_x} + 2\boldsymbol{a_y}\right)}{(2.24)^3}$$

$$dB_1 = 0.178 \times 10^{-10}\, \boldsymbol{a_z} \times (\boldsymbol{a_x} + 2\boldsymbol{a_y})$$

$$dB_1 = (-0.356\boldsymbol{a_x} + 0.178\boldsymbol{a_y}) \times 10^{-10} \quad T$$

$$d\,(\boldsymbol{dF_2}) = 4 \times 10^{-3}\, \boldsymbol{a_z} \times (-0.356\boldsymbol{a_x} + 0.178\boldsymbol{a_y}) \times 10^{-10}$$

$$d\,(\boldsymbol{dF_2}) = (-0.71\boldsymbol{a_x} - 1.42\boldsymbol{a_y}) \times 10^{-13}\, N$$

There is a force of attraction between the two conductors.

6-4 Force on a Current Carrying Loop

A rectangular current carrying loop is placed in a magnetic field as shown in Figure 6-8. The loop carries current in the counter clockwise direction and is located in $y = 0$ plane. $\boldsymbol{a}_x$, $\boldsymbol{a}_y$ and $\boldsymbol{a}_z$ are the unit vectors along x, y and $z - axis$ respectively. Force will act on the current carrying loop and we need to find out the total force on the loop. In order to find the total force acting on the current carrying loop, we find the force on side ab, side bc, side cd and side da. Vector sum of forces on the four sides of the loop results in the total force. For calculation of these forces, we recall the following equation.

$$\boldsymbol{F} = I\boldsymbol{L} \times \boldsymbol{B} \tag{6.13}$$

Force on side ab

$$\boldsymbol{F_{ab}} = I\boldsymbol{L} \times \boldsymbol{B} \tag{6.14}$$

Where $\boldsymbol{L} = -L\boldsymbol{a}_z$ and $\boldsymbol{B} = B\boldsymbol{a}_x$

Therefore $\boldsymbol{L} \times \boldsymbol{B} = -L\boldsymbol{a}_z \times B\boldsymbol{a}_x = -LB\boldsymbol{a}_y$

$$\boldsymbol{F_{ab}} = -ILB\boldsymbol{a}_y \tag{6.15}$$

Force on side bc

$$\boldsymbol{F_{bc}} = I\boldsymbol{L} \times \boldsymbol{B} \tag{6.16}$$

Where $\boldsymbol{L} = -W\boldsymbol{a}_x$ and $\boldsymbol{B} = B\boldsymbol{a}_x$

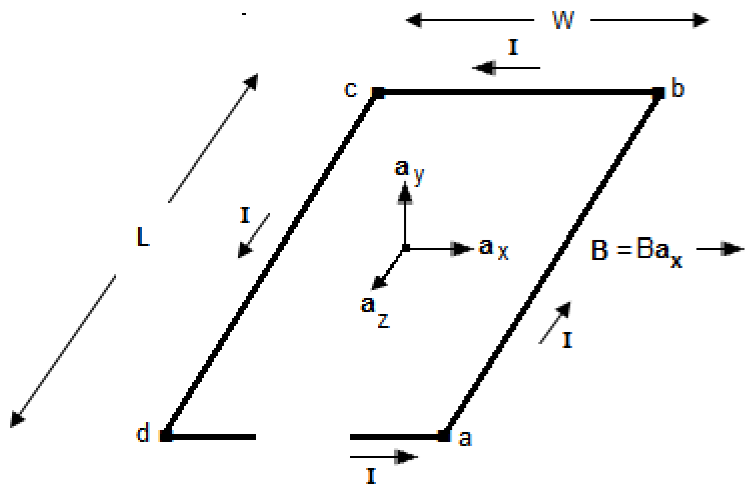


Figure 6-8: Current Carrying Loop

Therefore $\boldsymbol{L} \times \boldsymbol{B} = -W\boldsymbol{a}_x \times B\boldsymbol{a}_x = 0$

$$\boldsymbol{F}_{\boldsymbol{bc}} = -IW\boldsymbol{a}_x \times B\boldsymbol{a}_x = 0 \tag{6.17}$$

Force on side cd

$$\boldsymbol{F}_{\boldsymbol{cd}} = I\boldsymbol{L} \times \boldsymbol{B} \tag{6.18}$$

Where $\boldsymbol{L} = L\boldsymbol{a}_z$ and $\boldsymbol{B} = B\boldsymbol{a}_x$

Therefore $\boldsymbol{L} \times \boldsymbol{B} = L\boldsymbol{a}_z \times B\boldsymbol{a}_x = LB\boldsymbol{a}_y$

$$\boldsymbol{F}_{\boldsymbol{cd}} = ILB\boldsymbol{a}_y \tag{6.19}$$

Force on side da

$$\boldsymbol{F}_{\boldsymbol{da}} = I\boldsymbol{L} \times \boldsymbol{B} \tag{6.20}$$

Where $\boldsymbol{L} = W\boldsymbol{a}_x$ and $\boldsymbol{B} = B\boldsymbol{a}_x$

Therefore $\boldsymbol{L} \times \boldsymbol{B} = W\boldsymbol{a}_x \times B\boldsymbol{a}_x = 0$

$$\boldsymbol{F}_{\boldsymbol{da}} = IW\boldsymbol{a}_x \times B\boldsymbol{a}_x = 0 \tag{6.21}$$

Force on side ab is in the direction of negative y-axis, while force on side cd is in the direction of positive y-axis as shown in Figure 6-9. In presence of these two forces the current carrying loop will rotate in the clockwise direction around z-axis with a uniform angular velocity and will be in state of equilibrium.

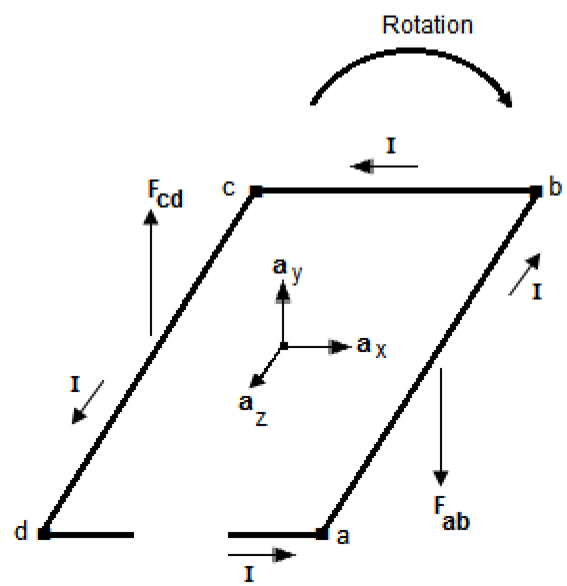


Figure 6-9: Rotation of Current Carrying Loop

The total force on the loop will be equal to the vector sum of all the four forces. That is

$$\boldsymbol{F} = \boldsymbol{F_{ab}} + \boldsymbol{F_{bc}} + \boldsymbol{F_{cd}} + \boldsymbol{F_{da}}$$

$$\boldsymbol{F} = -ILB\boldsymbol{a_y} + ILB\boldsymbol{a_y} = 0$$

Consider Figure 6-10 in which the width of the loop is visible. We want to find out the total torque on the loop.

Torque on side ab

$$\boldsymbol{T_{ab}} = \boldsymbol{Moment\ arm} \times \boldsymbol{F_{ab}}$$

$$\boldsymbol{T_{ab}} = \frac{w}{2}\boldsymbol{a_x} \times -ILB\boldsymbol{a_y}$$

$$\boldsymbol{T_{ab}} = -\frac{1}{2} wILB\boldsymbol{a_z}$$

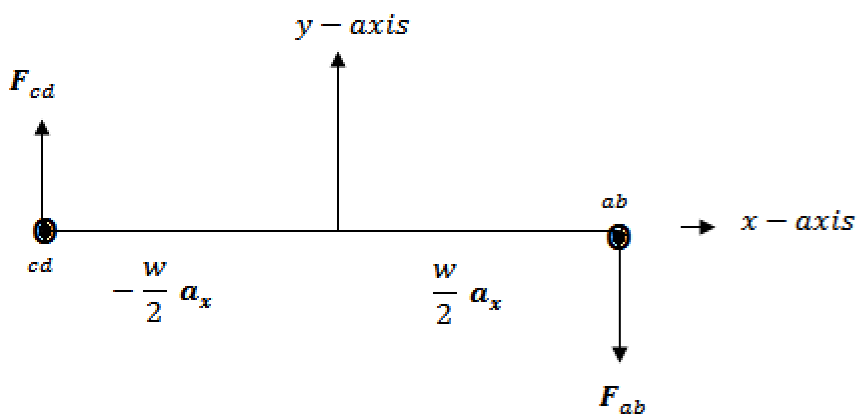


Figure 6-10: Torque on the Loop

Torque on side bc

$$\boldsymbol{T_{bc}} = \boldsymbol{Moment\ arm} \times \boldsymbol{F_{bc}}$$

$$\boldsymbol{T_{bc}} = \boldsymbol{Moment\ arm} \times 0$$

$$\boldsymbol{T_{bc}} = 0$$

Torque on side cd

$$\boldsymbol{T_{cd}} = \boldsymbol{Moment\ arm} \times \boldsymbol{F_{cd}}$$

$$\boldsymbol{T_{cd}} = -\frac{w}{2}\boldsymbol{a_x} \times ILB\boldsymbol{a_y}$$

$$\boldsymbol{T_{cd}} = -\frac{1}{2} wILB\boldsymbol{a_z}$$

Torque on side da

$$\boldsymbol{T_{da}} = \boldsymbol{Moment\ arm} \times \boldsymbol{F_{da}}$$

$$\boldsymbol{T_{da}} = \boldsymbol{Moment\ arm} \times 0$$

$$\boldsymbol{T_{da}} = 0$$

The total torque on the loop is given by

$$\boldsymbol{T} = \boldsymbol{T_{ab}} + \boldsymbol{T_{bc}} + \boldsymbol{T_{cd}} + \boldsymbol{T_{da}}$$

$$\boldsymbol{T} = -BILw\boldsymbol{a_z} \qquad (6.22)$$

The total torque can also be found as

$$\boldsymbol{T} = \boldsymbol{m} \times \boldsymbol{B} \qquad (6.23)$$

Where

$$\boldsymbol{m} = I\boldsymbol{S}$$

$$\boldsymbol{m} = ILw\boldsymbol{a_y}$$

Therefore

$$\boldsymbol{T} = ILw\boldsymbol{a_y} \times B\boldsymbol{a_x} = -BILw\boldsymbol{a_z}$$

Example 6-5:

Consider a current carrying loop in a uniform magnetic field of $2 \times 10^{-3}\ \boldsymbol{a_y}$ T as shown in Figure 6-11. Find total force and total torque on the loop if it carries a current of $4\ A$.

Solution:

Consider $\boldsymbol{L_1}$

$$\boldsymbol{F_1} = I\boldsymbol{L_1} \times \boldsymbol{B}$$

$$\boldsymbol{F_1} = 16\boldsymbol{a_x} \times 2 \times 10^{-3}\, \boldsymbol{a_y}$$

$$\boldsymbol{F_1} = 32\boldsymbol{a_z}\, mN$$

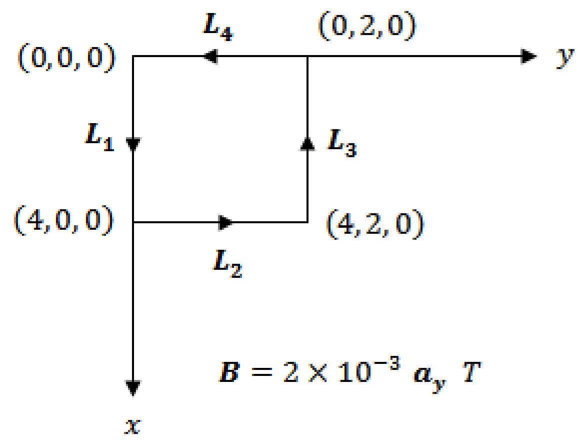


Figure 6-11 for Example 6-5

Consider $\boldsymbol{L_2}$

$$\boldsymbol{F_2} = I\boldsymbol{L_2} \times \boldsymbol{B}$$

$$\boldsymbol{F_2} = 8\boldsymbol{a_y} \times 2 \times 10^{-3}\, \boldsymbol{a_y}$$

$$\boldsymbol{F_2} = 0\ mN$$

Consider $\boldsymbol{L_3}$

$$\boldsymbol{F_3} = I\boldsymbol{L_3} \times \boldsymbol{B}$$

$$\boldsymbol{F_3} = -16\boldsymbol{a_x} \times 2 \times 10^{-3}\, \boldsymbol{a_y}$$

$$\boldsymbol{F_3} = -32\boldsymbol{a_z}\, mN$$

Consider $\boldsymbol{L_4}$

$$\boldsymbol{F_4} = I\boldsymbol{L_4} \times \boldsymbol{B}$$

$$\boldsymbol{F_4} = -8\boldsymbol{a_y} \times 2 \times 10^{-3}\, \boldsymbol{a_y}$$

$$\boldsymbol{F_4} = 0\ mN$$

$$\boldsymbol{F_T} = \boldsymbol{F_1}+\boldsymbol{F_2}+\boldsymbol{F_3}+\boldsymbol{F_4}$$
$$\boldsymbol{F_T} = 0 \;\; N$$

(b)

$$\boldsymbol{T} = I\boldsymbol{S} \times \boldsymbol{B}$$
$$\boldsymbol{T} = 4 \times 8\boldsymbol{a_z} \; \times 2 \times 10^{-3} \; \boldsymbol{a_y}$$
$$\boldsymbol{T} = 64\boldsymbol{a_x} \quad mN - m$$

Example 6-6:

Consider a current carrying loop in the magnetic field of an infinitely long current carrying as shown in Figure 6-12. Find total force on the loop if it carries a current of $1\ A$.

Solution:

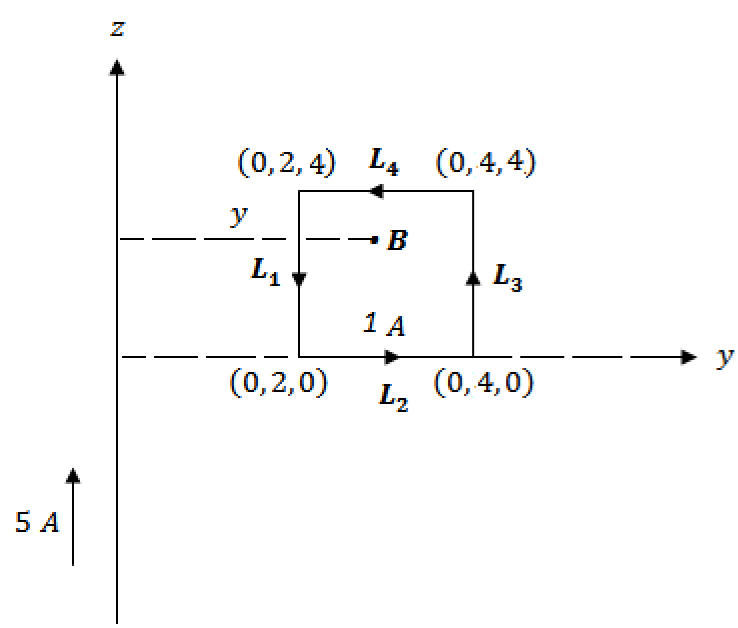


Figure 6-12: Loop in the magnetic field of current carrying conductor

Consider $\boldsymbol{H}$ at point B

$$\boldsymbol{H} = \frac{I}{2\pi\rho}\boldsymbol{a}_{\emptyset}$$

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi\rho}\boldsymbol{a}_{\emptyset}$$

$$\boldsymbol{B} = -\frac{\mu_0 I}{2\pi y}\boldsymbol{a}_{\boldsymbol{x}}$$

$$\boldsymbol{B} = -\frac{10^{-6}}{y}\boldsymbol{a}_{\boldsymbol{x}}$$

Consider L_1

$$x = 0, \qquad so \quad dx = 0$$

$$y = 2, \qquad so \quad dy = 0$$

$$0 \leq z \leq 4$$

$$\boldsymbol{d\ell} = dz\boldsymbol{a}_{\boldsymbol{z}}$$

$$\boldsymbol{F}_{\mathbf{1}} = I\int \boldsymbol{d\ell} \times \boldsymbol{B}$$

$$\boldsymbol{F}_{\mathbf{1}} = \int_{4}^{0} dz\boldsymbol{a}_{\boldsymbol{z}} \times -\frac{10^{-6}}{2}\boldsymbol{a}_{\boldsymbol{x}}$$

$$\boldsymbol{F}_{\mathbf{1}} = -0.5 \times 10^{-6} \times [z]_4^0\, \boldsymbol{a}_{\boldsymbol{y}}$$

$$\boldsymbol{F}_{\mathbf{1}} = 2 \times 10^{-6}\, \boldsymbol{a}_{\boldsymbol{y}} \qquad N$$

Consider L_2

$$x = 0, \qquad so \quad dx = 0$$

$$z = 0, \qquad so \quad dz = 0$$

$$2 \le y \le 4$$

$$\boldsymbol{d\ell} = dy\boldsymbol{a_y}$$

$$\boldsymbol{F_2} = I \int \boldsymbol{d\ell} \times \boldsymbol{B}$$

$$\boldsymbol{F_2} = \int_2^4 dy\boldsymbol{a_y} \times -\frac{10^{-6}}{y}\boldsymbol{a_x}$$

$$\boldsymbol{F_2} = 10^{-6} \times [\ln y]_2^4 \, \boldsymbol{a_z}$$

$$\boldsymbol{F_2} = 10^{-6} \times \ln\frac{4}{2} \, \boldsymbol{a_z}$$

$$\boldsymbol{F_2} = 0.69 \times 10^{-6} \, \boldsymbol{a_z} \, N$$

Consider L_3

$$x = 0, \qquad so \quad dx = 0$$

$$y = 4, \qquad so \quad dy = 0$$

$$0 \le z \le 4$$

$$\boldsymbol{d\ell} = dz\boldsymbol{a_z}$$

$$\boldsymbol{F_3} = I \int \boldsymbol{d\ell} \times \boldsymbol{B}$$

$$\boldsymbol{F_3} = \int_0^4 dz\boldsymbol{a_z} \times -\frac{10^{-6}}{4}\boldsymbol{a_x}$$

$$\boldsymbol{F_3} = -0.25 \times 10^{-6} \times \ [z]_0^4 \ \boldsymbol{a_y}$$

$$\boldsymbol{F_3} = -1 \times 10^{-6} \ \boldsymbol{a_y} \qquad N$$

Consider L_2

$$x = 0, \qquad so \quad dx = 0$$

$$z = 4, \qquad so \quad dz = 0$$

$$2 \le y \le 4$$

$$\boldsymbol{d\ell} = dy\boldsymbol{a_y}$$

$$\boldsymbol{F_4} = I \int \boldsymbol{d\ell} \times \boldsymbol{B}$$

$$\boldsymbol{F_4} = \int_4^2 dy\boldsymbol{a_y} \times -\frac{10^{-6}}{y}\boldsymbol{a_x}$$

$$\boldsymbol{F_4} = 10^{-6} \times [\ln y]_4^2 \ \boldsymbol{a_z}$$

$$\boldsymbol{F_4} = 10^{-6} \ \times \ \ln\frac{2}{4} \ \boldsymbol{a_z}$$

$$\boldsymbol{F_4} = -0.69 \times 10^{-6} \ \ \boldsymbol{a_z} \ \ N$$

$$\boldsymbol{F_T} = \boldsymbol{F_1}+\boldsymbol{F_2}+\boldsymbol{F_3}+\boldsymbol{F_4}$$

$$\boldsymbol{F_T} = 1 \times 10^{-6} \ \boldsymbol{a_y} \qquad N$$

Chapter 7

Maxwell's Equations

7-1 Maxwell's 4th Equation

A conductor is placed in a time varying magnetic field as shown in Figure 7-1. The variation in the strength of the magnetic field will induce some voltage across the conductor in accordance with Faraday's Law.

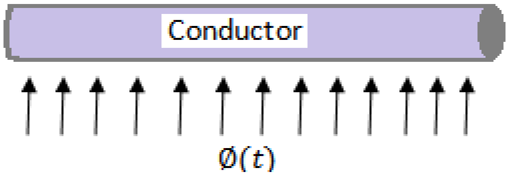


Figure 7-1: Conductor in a time varying magnetic flux

Voltage induced across the conductor in accordance with Faraday's law is given by

$$v = -\frac{d\emptyset}{dt}$$

The magnetic flux is given by

$$\emptyset = \int \boldsymbol{B.ds}$$

Therefore

$$v = \int -\frac{\partial \boldsymbol{B}}{\partial t}.\boldsymbol{ds} \qquad (7.1)$$

This voltage may be found in terms of electric field intensity

$$v = \oint \boldsymbol{E.d\ell} \qquad (7.2)$$

Applying Stoke's theorem

$$v = \int \nabla \times \boldsymbol{E}.\boldsymbol{ds} \qquad (7.3)$$

Therefore

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad (7.4)$$

This equation is known as Maxwell's 4th equation.

Consider a conductor of length d that slides on the rail in the direction of y-axis in a constant magnetic field as shown in Figure 7-2.

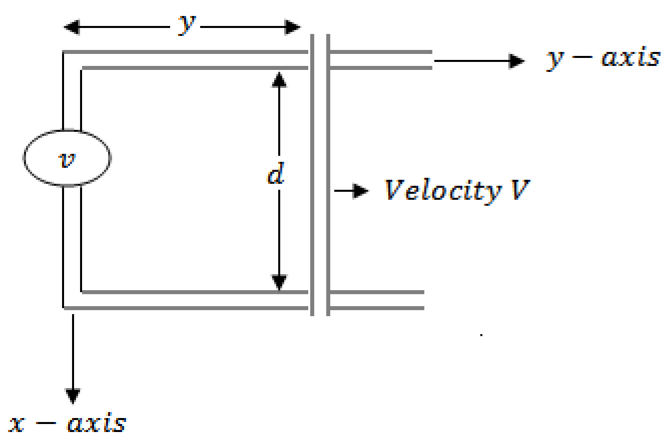


Figure 7-2: A sliding conductor in a constant magnetic field

The magnetic flux density is along $z - axis$. Voltage induced across the sliding conductor in accordance with Faraday's law is given by

$$v = -\frac{d\emptyset}{dt}$$

The flux linking the area is

$$\emptyset = Byd$$

Therefore

$$v = -B\frac{dy}{dt}\ d$$

Where $\frac{dy}{dt}$ represents the velocity of the sliding conductor. So the voltmeter reads

$$v = -BV\ d$$

Example 7.1:

The time varying magnetic flux in the vicinity of a conductor is $\emptyset = 5\cos 100\pi t \ \ m\,wb$. Find voltage induced in the conductor at $t = 1\, m\, sec$.

Solution:

$$v = -\frac{d\emptyset}{dt}$$

$$v = 5 \times 100\pi \ \times 10^{-3}\ \sin 100\pi t$$

$$v = 1.57\sin 100\pi t \quad V$$

$$v = 1.57\sin(100\pi \times 10^{-3})$$

$$v = 8.22\, mV$$

Example 7-2:

If $\boldsymbol{E} = 8\sin(6283t + 6z)\,\boldsymbol{a_x} \ \ V/m$. Find

$$-\frac{\partial \boldsymbol{B}}{\partial t}$$

Solution:

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$-\frac{\partial \boldsymbol{B}}{\partial t} = \begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$-\frac{\partial \boldsymbol{B}}{\partial t} = \begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 8\sin(6283t + 6z) & 0 & 0 \end{vmatrix}$$

$$-\frac{\partial \boldsymbol{B}}{\partial t} = 48\cos(6283t + 6z)\,\boldsymbol{a_y}$$

Example 7-3:

The sliding conductor in Figure 7-2 is moving with 16 m/sec in magnetic flux density of $4m\,T$. Find the reading of the voltmeter if length of the sliding conductor is $0.2m$.

Solution:

$$v = -BV\,d$$

$$v = -4 \times 10^{-3} \times 16 \times\ 0.2$$

$$v = -12.8\,m\,V$$

7-2 Equation of Continuity

Consider free charge of Q coulomb in a conductor as shown in Figure 7-3. We remove free electrons from the conductor in the outward direction. The free charge inside the conductor will decrease with respect to time.

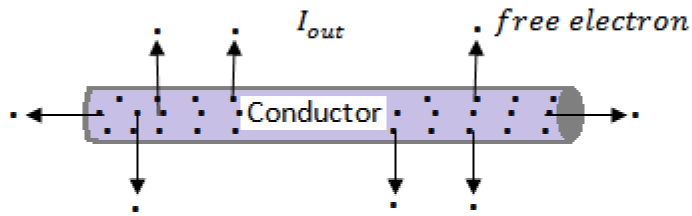


Figure 7-3: A conductor having free charge of Q coulomb

According to the law of conservation of charge, the rate of decrease of charge in the conductor will be equal to the current in the outward direction. Mathematically

$$I_{out} = -\frac{dQ}{dt} \tag{7.5}$$

The free charge inside the conductor is

$$Q = \int \rho_v \, dv$$

Differentiating both sides with respect to time, we obtain

$$\frac{dQ}{dt} = \int \frac{\partial \rho_v}{\partial t} dv$$

Therefore

$$I_{out} = \int \frac{-\partial \rho_v}{\partial t} dv \tag{7.6}$$

The current in the outward direction from the conductor is given by

$$I_{out} = \oint \boldsymbol{J.ds}$$

Applying Divergence theorem on the left hand side

$$I_{out} = \int (\nabla . \boldsymbol{J})dv$$

Therefore

$$\int (\nabla . \boldsymbol{J})dv = \int \frac{-\partial \rho_v}{\partial t} dv \qquad (7.7)$$

$$\nabla . \boldsymbol{J} = \frac{-\partial \rho_v}{\partial t} \qquad (7.8)$$

This equation is known as equation of continuity.

7-3 Maxwell's 3rd Equation

The Maxwell's 3rd equation is as under

$$(\nabla \times \boldsymbol{H}) = \boldsymbol{J} \qquad (7.9)$$

Consider a current carrying conductor extending from $-\infty$ to ∞ as shown in Figure 7-4.

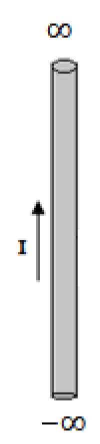


Figure 7-4: Current Carrying Conductor

The magnetic field intensity generated by the current in this conductor is given by

$$\boldsymbol{H} = \frac{I}{2\pi\rho}\,\boldsymbol{a}_\emptyset \qquad (\mathbf{7}.10)$$

The curl of $\boldsymbol{H}$ in cylindrical coordinate system is calculated as

$$\nabla \times \boldsymbol{H} = \frac{1}{\rho}\begin{vmatrix} \boldsymbol{a}_\rho & \rho\boldsymbol{a}_\emptyset & \boldsymbol{a}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\emptyset} & \dfrac{\partial}{\partial z} \\ H_\rho & \rho H_\emptyset & H_z \end{vmatrix}$$

$$\nabla \times \boldsymbol{H} = \begin{vmatrix} \boldsymbol{a}_\rho & \rho\boldsymbol{a}_\emptyset & \boldsymbol{a}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\emptyset} & \dfrac{\partial}{\partial z} \\ 0 & \dfrac{I}{2\pi\rho} & 0 \end{vmatrix}$$

$$\mathbf{A} = \nabla \times \boldsymbol{H} = -\frac{I}{2\pi\rho^2}\,\boldsymbol{a}_z \qquad (7.11)$$

The divergence of $\boldsymbol{A}$ is given by

$$\nabla.\boldsymbol{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\emptyset}{\partial\emptyset} + \frac{\partial A_z}{\partial z} = 0 \qquad (7.12)$$

$$\nabla.(\nabla \times \boldsymbol{H}) = \mathbf{0} \qquad (7.13)$$

Therefore

$$\nabla.\boldsymbol{J} = \mathbf{0} \qquad (7.14)$$

But the equation of continuity says that

$$\nabla.\boldsymbol{J} = \frac{-\partial\rho_v}{\partial t} \qquad (7.15)$$

This means that the Maxwell's 3rd equation is incorrect. Recall the Maxwell's first equation

$$\nabla.\boldsymbol{D} = \rho_v$$

$$\nabla.\boldsymbol{J} = -\frac{\partial}{\partial t}\,(\,\nabla.\boldsymbol{D}\,)$$

$$\nabla.\boldsymbol{J} = -\nabla.\frac{\partial \boldsymbol{D}}{\partial t}$$

$$\nabla.\boldsymbol{J} + \nabla.\frac{\partial \boldsymbol{D}}{\partial t} = 0$$

$$\nabla.\left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}\right) = 0 \qquad (7.16)$$

We compare equation 7.13 with equation 7.16

$$\nabla.(\nabla \times \boldsymbol{H}) = \nabla.\left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}\right)$$

$$(\nabla \times \boldsymbol{H}) = \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}\right) \qquad (7.17)$$

This is the correct version of Maxwell's 3rd equation, where

$$\frac{\partial \boldsymbol{D}}{\partial t} = \text{Displacement Current Density}$$

Example 7-4:

If $\boldsymbol{H} = 4\sin(6283t + 2z)\,\boldsymbol{a_x}\;\; A/m$. Find amplitude of the displacement current density if the conductivity is zero.

Solution:

$$(\nabla \times \boldsymbol{H}) = \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}\right)$$

$$(\nabla \times \boldsymbol{H}) = \left(\sigma\boldsymbol{E} + \frac{\partial \boldsymbol{D}}{\partial t}\right)$$

As

$$\sigma = 0$$

$$(\nabla \times \boldsymbol{H}) = \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\frac{\partial \boldsymbol{D}}{\partial t} = \begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4\sin(6283t + 2z) & 0 & 0 \end{vmatrix}$$

$$\frac{\partial \boldsymbol{D}}{\partial t} = 8\cos(6283t + 2z)\,\boldsymbol{a_y} \qquad A/m^2$$

7-4 Maxwell's Equations in Instantaneous Form

The first equation of Maxwell is given by

$$\boldsymbol{\nabla}.\boldsymbol{D} = \rho_v$$

The second equation of Maxwell is given by

$$\boldsymbol{\nabla}.\boldsymbol{B} = 0$$

The third equation of Maxwell is given by

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

The 4th equation of Maxwell is given by

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

7-5 Maxwell's Equations in Integral Form

Consider the Gauss's Law

$$\oint \boldsymbol{D}.\boldsymbol{ds} = \int \rho_v \; dv \qquad (7.18)$$

This is the integral form of the first equation of Maxwell. If we apply divergence theorem on the left hand side of this equation, then we obtain the first equation of Maxwell.

We know that the net magnetic flux passing through a closed surface is zero.

$$\oint \boldsymbol{B}.\boldsymbol{ds} = 0 \tag{7.19}$$

This is the integral form of the second equation of Maxwell. If we apply divergence theorem on the left hand side of this equation, then we obtain the second equation of Maxwell.

The third equation of Maxell's was derived in light of Ampere's circuital law which states that the integral of magnetic field intensity around a closed path is equal to the current enclosed by the closed path.

$$\oint \boldsymbol{H}.\boldsymbol{d\ell} = \int \boldsymbol{J}.\boldsymbol{ds} + \int \frac{\partial \boldsymbol{D}}{\partial t}.\boldsymbol{ds} \tag{7.20}$$

This is the integral form of the third equation of Maxwell. If we apply Stoke's theorem on the left hand side of this equation, then we obtain the third equation of Maxwell. The fourth equation of Maxell's was derived in light of Faraday's law which states that the if a conductor is placed in time varying magnetic flux, voltage is induced across the conductor.

$$\oint \boldsymbol{E}.\boldsymbol{d\ell} = \int -\frac{\partial \boldsymbol{B}}{\partial t}.\boldsymbol{ds} \tag{7.21}$$

This is the integral form of the fourth equation of Maxwell. If we apply Stoke's theorem on the left hand side of this equation, then we obtain fourth equation of Maxwell.

7-6 Maxwell's Equations in Phasor Form

Let us consider table 1 which shows the corresponding phasor values of different electrical quantities.

Table 7-1: Phasor Values

Instantaneous Value	Phasor Value
$\boldsymbol{E}$	$\vec{\boldsymbol{E}}$
$\boldsymbol{D}$	$\vec{\boldsymbol{D}}$
$\boldsymbol{H}$	$\vec{\boldsymbol{H}}$
$\boldsymbol{B}$	$\vec{\boldsymbol{B}}$
$\boldsymbol{J}$	$\vec{\boldsymbol{J}}$

In order to convert instantaneous form of Maxwell's equations to phasor form, we need to replace the instantaneous values of the electrical quantities given in Table 1 by the phasor values and the $\frac{\partial}{\partial t}$ by $j\omega$.

The first equation of Maxwell is given by

$$\boldsymbol{\nabla}.\boldsymbol{D} = \rho_v$$

The same equation in phasor form is given by

$$\boldsymbol{\nabla}.\vec{\boldsymbol{D}} = \rho_v \qquad (7.22)$$

The second equation of Maxwell is given by

$$\boldsymbol{\nabla}.\boldsymbol{B} = 0$$

The same equation in phasor form is given by

$$\boldsymbol{\nabla}.\vec{\boldsymbol{B}} = 0 \qquad (7.23)$$

The third equation of Maxwell is given by

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

The same equation in phasor form is given by

$$\boldsymbol{\nabla} \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}} + j\omega\,\vec{\boldsymbol{D}} \qquad (7.24)$$

The 4th equation of Maxwell is given by

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

The same equation in phasor form is given by

$$\nabla \times \overrightarrow{\boldsymbol{E}} = -j\omega \overrightarrow{\boldsymbol{B}} \qquad (7.25)$$

Chapter 8

Electromagnetic Waves

8-1 Propagation of TEM wave in Lossy Dielectric Medium

Transverse electromagnetic waves are also known as uniform plane wave. In transverse electromagnetic wave, the electric field intensity, magnetic field intensity and velocity are normal to one another as shown in Figure 8-1. Lossy medium is a practical medium and the attenuation of the signal takes place in this medium. The conductivity (σ) of the medium is not zero.

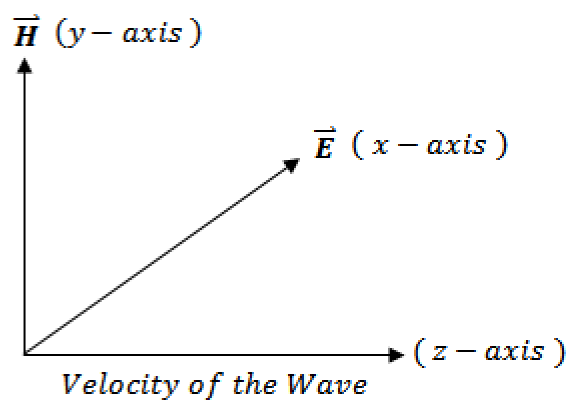


Figure 8-1: TEM Wave Propagating in z Direction

In order to find out the electric field intensity of the TEM wave, we consider the following Maxwell's equation in phasor form

$$\nabla \times \vec{E} = -j\omega \vec{B} \quad (8.1)$$

$$\nabla \times \vec{E} = -j\omega \mu\vec{H} \quad (8.2)$$

$$\nabla \times (\nabla \times \vec{E}) = -j\omega \mu (\nabla \times \vec{H}) \quad (8.3)$$

We recall another equation of Maxwell's

$$\nabla \times \vec{H} = \vec{J} + j\omega \vec{D} \quad (8.4)$$

Therefore

$$\nabla \times \left(\nabla \times \vec{\boldsymbol{E}}\right) = -j\omega\,\mu\,(\vec{\boldsymbol{J}} \;+ j\omega\,\vec{\boldsymbol{D}}\,) \qquad (8.5)$$

$$\nabla \times \left(\nabla \times \vec{\boldsymbol{E}}\right) = -j\omega\,\mu\,(\vec{\boldsymbol{J}} \;+ j\omega\varepsilon\,\vec{\boldsymbol{E}}\,) \qquad (8.6)$$

$$\nabla \times \left(\nabla \times \vec{\boldsymbol{E}}\right) = -j\omega\,\mu\,(\sigma\,\vec{\boldsymbol{E}} \;+ j\omega\varepsilon\,\vec{\boldsymbol{E}}\,) \qquad (8.7)$$

$$\nabla \times \left(\nabla \times \vec{\boldsymbol{E}}\right) = -j\omega\,\mu\,\sigma\,\vec{\boldsymbol{E}} + \omega^2\,\mu\varepsilon\,\vec{\boldsymbol{E}} \qquad (8.8)$$

It can be shown that

$$\nabla \times \left(\nabla \times \vec{\boldsymbol{E}}\right) = -\nabla^2\vec{\boldsymbol{E}}$$

Where

$$\nabla^2\vec{\boldsymbol{E}} = \frac{\partial^2\vec{\boldsymbol{E}}}{\partial x^2} + \frac{\partial^2\vec{\boldsymbol{E}}}{\partial y^2} + \frac{\partial^2\vec{\boldsymbol{E}}}{\partial z^2}$$

$$-\nabla^2\vec{\boldsymbol{E}} = -j\omega\,\mu\,\sigma\,\vec{\boldsymbol{E}} + \omega^2\,\mu\varepsilon\,\vec{\boldsymbol{E}} \qquad (8.9)$$

$$\nabla^2\vec{\boldsymbol{E}} \;+ \omega^2\,\mu\varepsilon\,\vec{\boldsymbol{E}} - \;j\omega\,\mu\,\sigma\,\vec{\boldsymbol{E}} = 0 \qquad (8.10)$$

$$\nabla^2\vec{\boldsymbol{E}} \;+ \omega^2\,\mu\,\varepsilon\,\left(1 - j\,\frac{\sigma}{\omega\,\varepsilon}\right)\vec{\boldsymbol{E}} = \;0 \qquad (8.11)$$

Let

$$\hat{\varepsilon} = \varepsilon\,\left(1 - j\,\frac{\sigma}{\omega\,\varepsilon}\right)$$

$$\nabla^2\vec{\boldsymbol{E}} \;+ \omega^2\,\mu\,\hat{\varepsilon}\,\vec{\boldsymbol{E}} = \;0 \qquad (8.12)$$

This is the wave equation in a lossy dielectric medium. As the TEM wave is travelling along $z - axis$, the energy of the wave will decrease with respect to z and z represents the distance travelled by the wave.

$$E = f(z)$$

Therefore

$$\nabla^2\vec{\boldsymbol{E}} = \frac{\partial^2\vec{\boldsymbol{E}}}{\partial z^2}$$

$$\frac{\partial^2 \vec{\boldsymbol{E}}}{\partial z^2} + \omega^2 \mu \hat{\varepsilon} \vec{\boldsymbol{E}} = 0 \tag{8.13}$$

This is 2nd order homogeneous differential equation. In order to find its solution, we need to find its characteristic or auxiliary equation. Let

$$\frac{\partial}{\partial z} = m$$

$$m^2 \vec{\boldsymbol{E}} + \omega^2 \mu \hat{\varepsilon} \vec{\boldsymbol{E}} = 0 \tag{8.14}$$

$$(m^2 + \omega^2 \mu \hat{\varepsilon}) \vec{\boldsymbol{E}} = 0 \tag{8.15}$$

$$(m^2 + \omega^2 \mu \hat{\varepsilon}) = 0 \tag{8.16}$$

This is the characteristic or auxiliary equation of the 2nd order homogeneous differential equation. The roots of the auxiliary equation are complex

$$m = \pm j\omega\sqrt{\mu\hat{\varepsilon}} \tag{8.17}$$

$$m = \pm \hat{\gamma} = \pm (\alpha + j\beta)$$

$\hat{\gamma}$ is known as propagation constant, α is known as attenuation constant and β is known as phase constant. The solution of the 2nd order homogeneous differential equation is

$$\vec{\boldsymbol{E}} = \left(E_0 \, e^{-\hat{\gamma} z} + E_b \, e^{\hat{\gamma} z} \right) \boldsymbol{a_x} \tag{8.18}$$

The first component on the right hand side represents the component of the wave travelling along positive $z - axis$ and the second component represents the component of the wave travelling along negative $z - axis$. The first component is known as forward travelling wave and the second component is known as backward travelling wave. No component of the wave is travelling along negative $z - axis$, so we do ignore this component. When the reflection of the wave takes place due to change in the medium, then this component exists.

$$\vec{\boldsymbol{E}} = \left(E_0 \, e^{-\hat{\gamma} z} \right) \boldsymbol{a_x} \tag{8.19}$$

$$\vec{\boldsymbol{E}} = \left(E_0 \, e^{-\alpha z - j\beta z} \right) \boldsymbol{a_x} \tag{8.20}$$

This is the phasor value of the electric field intensity of the uniform plane wave, propagating in a lossy dielectric medium. Its instantaneous equation is given by

$$\boldsymbol{E} = E_0 \, e^{-\alpha z} \, \sin(\omega t - \beta z)\boldsymbol{a_x}$$

8-2 Magnetic Field Intensity of TEM wave in Lossy Dielectric Medium

Consider the phasor value of the electric field intensity of the uniform plane wave, propagating in a lossy dielectric medium

$$\vec{\boldsymbol{E}} = \left(E_0 \, e^{-\hat{\gamma} z} \right) \boldsymbol{a_x} \tag{8.21}$$

Lets us determine Magnetic Field Intensity of TEM waves in this medium with the help of

$$\boldsymbol{\nabla} \times \vec{\boldsymbol{E}} = -j\omega \, \mu \vec{\boldsymbol{H}} \tag{8.22}$$

$$\vec{\boldsymbol{H}} = \frac{1}{-j\omega \, \mu} \left(\boldsymbol{\nabla} \times \vec{\boldsymbol{E}} \right) \tag{8.23}$$

$$\vec{\boldsymbol{H}} = \frac{1}{-j\omega \, \mu} \begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$\vec{\boldsymbol{H}} = \frac{1}{-j\omega \, \mu} \begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_0 \, e^{-\hat{\gamma} z} & 0 & 0 \end{vmatrix}$$

$$\vec{\boldsymbol{H}} = \frac{-\hat{\gamma}}{-j\omega \, \mu} \, E_0 \, e^{-\hat{\gamma} z} \, \boldsymbol{a_y}$$

$$\vec{\boldsymbol{H}} = \frac{-j\omega\sqrt{\mu\hat{\varepsilon}}}{-j\omega\,\mu}\, E_0\, e^{-\hat{\gamma}\,z}\, \boldsymbol{a_y}$$

$$\vec{\boldsymbol{H}} = \frac{1}{\sqrt{\frac{\mu}{\hat{\varepsilon}}}}\, E_0\, e^{-\hat{\gamma}\,z}\, \boldsymbol{a_y} \qquad (8.24)$$

Where

$$\hat{\eta} = \sqrt{\frac{\mu}{\hat{\varepsilon}}} = \eta\, e^{j\theta}$$

is the characteristic or intrinsic impedance of the lossy dielectric medium.

$$\vec{\boldsymbol{H}} = \frac{1}{\eta\, e^{j\theta}}\, (\, E_0\, e^{-\alpha z - j\beta z}\,)\, \boldsymbol{a_y} \qquad (8.25)$$

$$\vec{\boldsymbol{H}} = \frac{1}{\eta}\, (\, E_0\, e^{-\alpha z - j\beta z - j\theta}\,)\, \boldsymbol{a_y} \qquad (8.26)$$

This is the phasor value of the magnetic field intensity of the uniform plane wave, propagating in a lossy dielectric medium. Its instantaneous equation is given by

$$\boldsymbol{H} = \frac{E_0}{\eta}\, e^{-\alpha z}\, \sin(\omega t - \beta z - \theta)\boldsymbol{a_y}$$

The average power density of the wave is given by

$$\boldsymbol{S_{ave}} = \frac{1}{2}\, Real\, (\, \vec{\boldsymbol{E}} \times \vec{\boldsymbol{H^*}}\,)$$

Where $\vec{\boldsymbol{H^*}}$ is the conjugate of $\vec{\boldsymbol{H}}$

Example 8-1

$\boldsymbol{E} = 0.1\, e^{-\alpha z}\, \sin(\omega t - \beta z)\boldsymbol{a_x}$ is travelling in a medium characterized by $\boldsymbol{\varepsilon_r} = 2.5,\ \delta = 2.5\ and\ \mu_r = 2.5$, if frequency of the wave is $1.8GHz$, then find $\boldsymbol{E},\ \boldsymbol{H}\ ,\ v\ and\ \lambda.$

Answer

$$\omega = 2\,\pi \times 1.8 \times 10^9 = 1.13 \times 10^{10}\ rad/sec$$

$$\alpha + j\beta \;=\; j\omega\sqrt{\mu\hat{\varepsilon}}$$

$$\alpha + j\beta \;=\; j\omega\sqrt{\mu\varepsilon\left(1 - j\,\frac{\sigma}{\omega\,\varepsilon}\right)}$$

$$=\; j1.13\times10^{10}\sqrt{1.6\times4\,\pi\times10^{-7}}\times\sqrt{(2.5\times8.85\times10^{-12} - j\,\frac{2.5}{1.13\times10^{10}})}$$

$$\alpha + j\beta \;=\; 156.85 + j179.55$$

$$\boldsymbol{E} = \;0.1\,e^{-156.85z}\;\sin(1.813\times10^{10}t - 179.55z)\boldsymbol{a_x}\;V/m$$

$$v = \frac{\omega}{\beta}$$

$$v = \frac{1.13\times10^{10}}{179.55}$$

$$v = 6.3\times10^{7}\;m/sec$$

$$\lambda = \frac{2\,\pi}{\beta}$$

$$\lambda = \frac{2\,\pi}{179.55}$$

$$\lambda = 34.99mm$$

$$\hat{\eta} = \sqrt{\frac{\mu}{\left(\varepsilon - j\,\frac{\sigma}{\omega}\right)}}$$

$$\hat{\eta} = \sqrt{\frac{1.6\times4\,\pi\times10^{-7}}{\left(2.5\times8.85\times10^{-12} - j\,\frac{2.5}{1.13\times10^{10}}\right)}}$$

$$\hat{\eta} = 95.29\angle41.14\;\Omega$$

$$\boldsymbol{H} = \frac{E_0}{\eta}\;e^{-\alpha z}\;\sin(\omega t - \beta z - \theta)\boldsymbol{a_y}$$

$$\boldsymbol{H} = 0.001\, e^{-156.85z} \,\sin(1.813 \times 10^{10} t - 179.55z - 41.14^0)\boldsymbol{a_y}$$

Question 8-2

$\boldsymbol{E} = 0.1\, e^{-\alpha z} \,\sin(\omega t - \beta z)\boldsymbol{a_x}$ is travelling in a medium characterized by $\boldsymbol{\varepsilon_r} = 2.5, \delta = 2.5$ and $\mu_r = 2.5$, if frequency of the wave is $1.8KHz$, then find $\boldsymbol{E}$, $\boldsymbol{H}$, and λ.

8-3 Propagation of TEM wave in Lossless Dielectric Medium

Transverse electromagnetic waves are also known as uniform plane wave. As discussed in the preceding sections the electric field intensity, magnetic field intensity and velocity of transverse electromagnetic wave, are normal to one another as shown in Figure 8-2. Lossless medium is an ideal medium and the attenuation of the signal does not take place in this medium. The conductivity (σ) of the medium is zero.

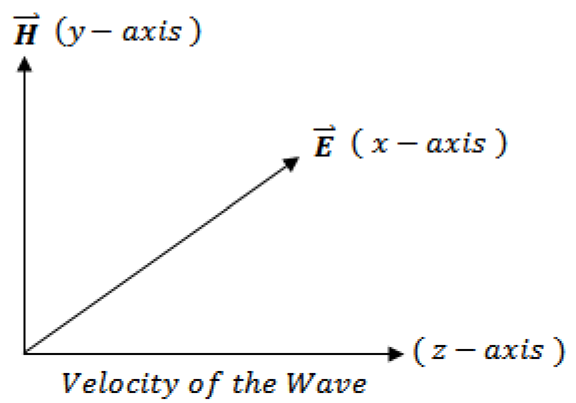


Figure 8-2: TEM Wave Propagating in z Direction

In order to find out the electric field intensity of the TEM wave, we consider the following Maxwell's equation in phasor form

$$\nabla \times \vec{\boldsymbol{E}} = -j\omega\, \vec{\boldsymbol{B}} \qquad (8.27)$$

$$\nabla \times \vec{\boldsymbol{E}} = -j\omega\, \mu\vec{\boldsymbol{H}} \qquad (8.28)$$

$$\nabla \times \left(\nabla \times \vec{\boldsymbol{E}}\right) = -j\omega\, \mu\, (\nabla \times \vec{\boldsymbol{H}}) \qquad (8.29)$$

We recall another equation of Maxwell's

$$\nabla \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}} + j\omega\, \vec{\boldsymbol{D}} \tag{8.30}$$

Therefore

$$\nabla \times \left(\nabla \times \vec{\boldsymbol{E}}\right) = -j\omega\, \mu\, (\vec{\boldsymbol{J}} + j\omega\, \vec{\boldsymbol{D}}) \tag{8.31}$$

$$\nabla \times \left(\nabla \times \vec{\boldsymbol{E}}\right) = -j\omega\, \mu\, (\vec{\boldsymbol{J}} + j\omega\varepsilon\, \vec{\boldsymbol{E}}) \tag{8.32}$$

$$\nabla \times \left(\nabla \times \vec{\boldsymbol{E}}\right) = -j\omega\, \mu\, (\sigma\, \vec{\boldsymbol{E}} + j\omega\varepsilon\, \vec{\boldsymbol{E}}) \tag{8.33}$$

As

$$\sigma = 0$$

$$\nabla \times \left(\nabla \times \vec{\boldsymbol{E}}\right) = -\omega^2\, \mu\varepsilon\, \vec{\boldsymbol{E}} \tag{8.34}$$

It can be shown that

$$\nabla \times \left(\nabla \times \vec{\boldsymbol{E}}\right) = -\nabla^2 \vec{\boldsymbol{E}}$$

Where

$$\nabla^2 \vec{\boldsymbol{E}} = \frac{\partial^2 \vec{\boldsymbol{E}}}{\partial x^2} + \frac{\partial^2 \vec{\boldsymbol{E}}}{\partial y^2} + \frac{\partial^2 \vec{\boldsymbol{E}}}{\partial z^2}$$

$$-\nabla^2 \vec{\boldsymbol{E}} = \omega^2\, \mu\varepsilon\, \vec{\boldsymbol{E}} \tag{8.35}$$

$$\nabla^2 \vec{\boldsymbol{E}} + \omega^2\, \mu\varepsilon\, \vec{\boldsymbol{E}} = 0 \tag{8.36}$$

This is the wave equation in a lossless dielectric medium. As the TEM wave is travelling along $z - axis$, the energy of the wave will change with respect to z and z represents the distance travelled by the wave.

$$E = f(z)$$

Therefore

$$\nabla^2 \vec{\boldsymbol{E}} = \frac{\partial^2 \vec{\boldsymbol{E}}}{\partial z^2}$$

$$\frac{\partial^2 \vec{\boldsymbol{E}}}{\partial z^2} + \omega^2 \mu \varepsilon \vec{\boldsymbol{E}} = 0 \tag{8.37}$$

This is 2nd order homogeneous differential equation. In order to find its solution, we need to find its characteristic or auxiliary equation. Let

$$\frac{\partial}{\partial z} = m$$

$$m^2 \vec{\boldsymbol{E}} + \omega^2 \mu \varepsilon \vec{\boldsymbol{E}} = 0 \tag{8.38}$$

$$(m^2 + \omega^2 \mu \varepsilon) \vec{\boldsymbol{E}} = 0 \tag{8.39}$$

$$(m^2 + \omega^2 \mu \varepsilon) = 0 \tag{8.40}$$

This is the characteristic or auxiliary equation of the 2nd order homogeneous differential equation. The roots of the auxiliary equation are imaginary

$$m = \pm j\omega\sqrt{\mu\varepsilon} \tag{8.41}$$

As

$$m = \pm (\alpha + j\beta)$$

So

$$\alpha = 0$$

And

$$\beta = \omega\sqrt{\mu\varepsilon}$$

The solution of the 2nd order homogeneous differential equation is

$$\vec{\boldsymbol{E}} = \left(E_0\, e^{-j\beta z} + E_b\, e^{\,j\beta z} \right) \boldsymbol{a_x} \tag{8.42}$$

Once again, the first component on the right hand side represents the component of the wave travelling along positive $z - axis$ and the second component represents the component of the wave travelling along negative $z - axis$. The first component is known as forward travelling wave and the second component is known as backward travelling wave. No component of the wave is travelling along negative $z - axis$, so we do ignore this component. When the reflection of the wave takes place due to change in the medium, then this component exists.

$$\vec{\boldsymbol{E}} = \left(E_0 \, e^{-j\beta \, z} \right) \boldsymbol{a_x} \tag{8.43}$$

This is the phasor value of the electric field intensity of the uniform plane wave, propagating in a lossless dielectric medium. Its instantaneous equation is given by

$$\boldsymbol{E} = E_0 \ \sin(\omega t - \beta z) \boldsymbol{a_x}$$

8-4 Magnetic Field Intensity of TEM wave in Lossless Medium

Consider the phasor value of the electric field intensity of the uniform plane wave, propagating in a lossless dielectric medium

$$\vec{\boldsymbol{E}} = \left(E_0 \, e^{-j\beta z} \right) \boldsymbol{a_x} \tag{8.44}$$

Lets us determine Magnetic Field Intensity of TEM waves in this medium with the help of

$$\boldsymbol{\nabla} \times \vec{\boldsymbol{E}} = -j\omega \, \mu \vec{\boldsymbol{H}} \tag{8.45}$$

$$\vec{\boldsymbol{H}} = \frac{1}{-j\omega \, \mu} \ \left(\boldsymbol{\nabla} \times \vec{\boldsymbol{E}} \right) \tag{8.46}$$

$$\vec{\boldsymbol{H}} = \frac{1}{-j\omega \, \mu} \begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$\vec{\boldsymbol{H}} = \frac{1}{-j\omega\,\mu}\begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_0\, e^{-j\beta z} & 0 & 0 \end{vmatrix}$$

$$\vec{\boldsymbol{H}} = \frac{-j\beta}{-j\omega\,\mu}\, E_0\, e^{-j\beta\, z}\, \boldsymbol{a_y}$$

$$\vec{\boldsymbol{H}} = \frac{-j\omega\sqrt{\mu\varepsilon}}{-j\omega\,\mu}\, E_0\, e^{-j\beta\, z}\, \boldsymbol{a_y}$$

$$\vec{\boldsymbol{H}} = \frac{1}{\sqrt{\frac{\mu}{\varepsilon}}}\, E_0\, e^{-j\beta\, z}\, \boldsymbol{a_y} \qquad (8.47)$$

Where

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

is the characteristic or intrinsic impedance of the lossless dielectric medium.

$$\vec{\boldsymbol{H}} = \frac{1}{\eta}\,(\,E_0\, e^{-j\beta z}\,)\,\boldsymbol{a_y} \qquad (8.48)$$

This is the phasor value of the magnetic field intensity of the uniform plane wave, propagating in a lossless dielectric medium. Its instantaneous equation is given by

$$\boldsymbol{H} = \frac{E_0}{\eta}\,\sin(\omega t - \beta z)\boldsymbol{a_y}$$

The average power density of the wave is given by

$$\boldsymbol{S_{ave}} = \frac{1}{2}\, Real\,(\,\vec{\boldsymbol{E}} \times \vec{\boldsymbol{H}^{*}}\,)$$

8-5 The Angular Frequency of TEM wave in a Dielectric Medium

The magnitude of the electric field intensity in a lossless dielectric medium is given by

$$E = E_0 \sin(\omega t - \beta z) \tag{8.49}$$

The intensity varies with time as well as distance. Let us study the variation in the intensity with respect to time only.

Let

$$\beta z = 0$$

Therefore

$$E = E_0 \sin(\omega t) \tag{8.50}$$

The waveform for the intensity as a function of time is shown in Figure 8-3.

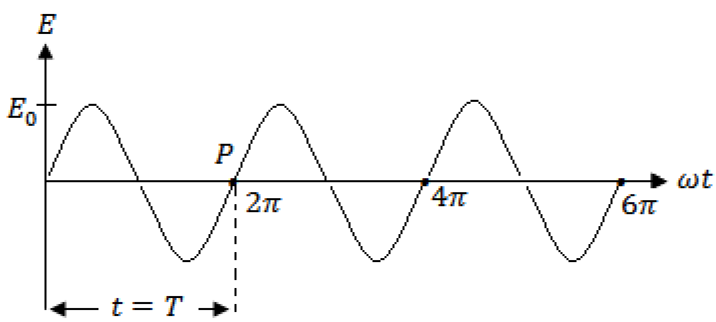


Figure 8-3: Variation in Intensity with respect to time

At point P

$$\omega t = 2\pi$$

The time taken by one complete cycle of the wave is defined as time period. Therefore

$$\omega T = 2\pi$$

$$\omega = \frac{2\pi}{T}$$

$$\omega = 2\pi f$$

8-6 The Phase Constant of TEM wave in a Dielectric Medium

The magnitude of the electric field intensity in a lossless dielectric medium is given by

$$E = E_0 \ \sin(\omega t - \beta z) \qquad (8.51)$$

The intensity varies with time as well as distance. Let us study the variation in the intensity with respect to distance only.

Let

$$\omega t = \pi$$

Therefore

$$E = E_0 \ \sin(\beta z)$$

The waveform for the intensity as a function of distance is shown in Figure 8-4.

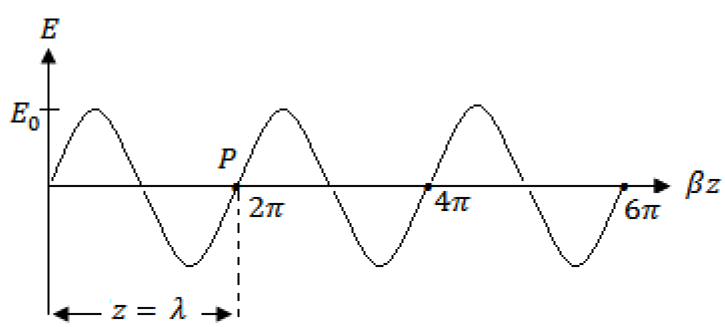


Figure 8-4: Variation in Intensity with respect to distance

At point P

$$\beta z = 2\pi$$

The distance travelled by one complete cycle of the wave is defined as wavelength. Therefore

$$\beta \lambda = 2\pi$$

$$\beta = \frac{2\pi}{\lambda}$$

8-7 Speed of TEM wave in a Dielectric Medium

The magnitude of the electric field intensity in a lossless dielectric medium is given by

$$E = E_0 \sin(\omega t - \beta z) \qquad (8.52)$$

$(\omega t - \beta z)$ is known as the phase of the wave. It can be shown that the phase of the wave is always constant. That is

$$\omega t - \beta z = Constant$$

Let us differentiate it with respect to time.

$$\omega - \beta \frac{dz}{dt} = 0$$

$$\frac{dz}{dt} = \frac{\omega}{\beta}$$

$$v = \frac{\omega}{\beta} \qquad (8.53)$$

As

$$\beta = \omega\sqrt{\mu\varepsilon}$$

Therefore

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \qquad (8.54)$$

This shows that the permeability and permittivity of the dielectric medium limits the speed of electromagnetic wave.

And as

$$\omega = 2\pi f$$

And

$$\beta = \frac{2\pi}{\lambda}$$

Therefore

$$v = f\lambda \qquad (8.55)$$

Example 8-3

$\boldsymbol{E} = 377 \ \sin(10^9\, t - 5y)\boldsymbol{a_z}$ is travelling in a medium characterized by $\mu = \mu_0$ find $\boldsymbol{\varepsilon_r}, \boldsymbol{H}\, , v \ \ \lambda\, and$ average power density**.**

Answer

$$v = \frac{\omega}{\beta}$$

$$v = \frac{10^9}{5}$$

$$v = 2 \times 10^8 \ m/sec$$

$$v = \frac{1}{\sqrt{\mu_0 \mu_r \varepsilon_0 \varepsilon_r}}$$

$$\varepsilon_r = (\frac{3 \times 10^8}{2 \times 10^8})^2$$

$$\varepsilon_r = 2.25$$

$$\lambda = \frac{2\pi}{\beta}$$

$$\lambda = \frac{2\pi}{5}$$

$$\lambda = 1.257\, m$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\eta = \sqrt{\frac{4\,\pi \times 10^{-7}}{2.25 \times 8.85 \times 10^{-12}}}$$

$$\eta = 251.33\Omega$$

$$\boldsymbol{H} = \frac{E_0}{\eta}\ \sin(\omega t - \beta z)\boldsymbol{a_y}$$

$$\boldsymbol{H} = \ 1.5\ \sin(10^9\, t - 5y)\boldsymbol{a_y}\ A/m$$

$$\boldsymbol{S_{av}} = \ 282.75\boldsymbol{a_y}\ w/m^2$$

8-8 Propagation of TEM wave in Free Space

Consider a uniform electromagnetic wave as shown in Figure 8-5. Free Space is a special lossless medium and the attenuation of the signal does not take place in this medium as well. The conductivity (σ) of the medium is zero. The permeability and the permittivity of this medium are represented by μ_0 and ε_0 .

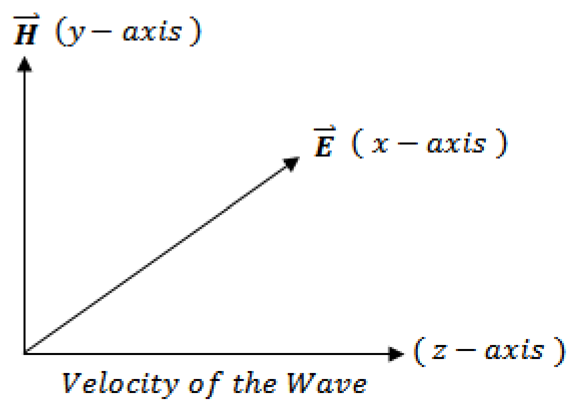


Figure 8-5: TEM Wave Propagating in Free Space

In order to find out the electric field intensity of the TEM wave, we consider the following Maxwell's equation in phasor form

$$\nabla \times \vec{\boldsymbol{E}} = -j\omega\, \vec{\boldsymbol{B}} \qquad (8.56)$$

$$\nabla \times \vec{\boldsymbol{E}} = -j\omega\, \mu_0 \vec{\boldsymbol{H}} \qquad (8.57)$$

$$\nabla \times \left(\nabla \times \overrightarrow{\boldsymbol{E}}\right) = -j\omega\, \mu_0\, (\, \nabla \times \overrightarrow{\boldsymbol{H}}\,) \qquad (8.58)$$

We recall another equation of Maxwell's

$$\nabla \times \overrightarrow{\boldsymbol{H}} = \vec{\boldsymbol{J}} \; + j\omega\, \overrightarrow{\boldsymbol{D}} \qquad (8.59)$$

Therefore

$$\nabla \times \left(\nabla \times \overrightarrow{\boldsymbol{E}}\right) = -j\omega\, \mu_0\, (\vec{\boldsymbol{J}} \; + j\omega\, \overrightarrow{\boldsymbol{D}} \qquad (8.60)$$

$$\nabla \times \left(\nabla \times \overrightarrow{\boldsymbol{E}}\right) = -j\omega\, \mu_0\, (\vec{\boldsymbol{J}} \; + j\omega\varepsilon_0\, \overrightarrow{\boldsymbol{E}}\,) \qquad (8.61)$$

$$\nabla \times \left(\nabla \times \overrightarrow{\boldsymbol{E}}\right) = -j\omega\, \mu_0\, (\sigma\, \overrightarrow{\boldsymbol{E}} \; + j\omega\varepsilon_0\, \overrightarrow{\boldsymbol{E}}\,) \qquad (8.62)$$

As

$$\sigma = 0$$

$$\nabla \times \left(\nabla \times \overrightarrow{\boldsymbol{E}}\right) = \omega^2\, \mu_0\varepsilon_0\, \overrightarrow{\boldsymbol{E}} \qquad (8.63)$$

It can be shown that

$$\nabla \times \left(\nabla \times \overrightarrow{\boldsymbol{E}}\right) = \; -\nabla^2\overrightarrow{\boldsymbol{E}} \qquad (8.64)$$

Where

$$\nabla^2\overrightarrow{\boldsymbol{E}} = \frac{\partial^2\overrightarrow{\boldsymbol{E}}}{\partial x^2} + \frac{\partial^2\overrightarrow{\boldsymbol{E}}}{\partial y^2} + \frac{\partial^2\overrightarrow{\boldsymbol{E}}}{\partial z^2}$$

$$-\nabla^2\overrightarrow{\boldsymbol{E}} = \; \omega^2\, \mu_0\varepsilon_0\, \overrightarrow{\boldsymbol{E}} \qquad (8.65)$$

$$\nabla^2\overrightarrow{\boldsymbol{E}} \; + \omega^2\, \mu_0\varepsilon_0\, \overrightarrow{\boldsymbol{E}} = 0 \qquad (8.66)$$

This is the wave equation in free space. As the TEM wave is travelling along $z - axis$, the energy of the wave will change with respect to z and z represents the distance travelled by the wave.

$$E = f(z)$$

Therefore

$$\nabla^2 \vec{\boldsymbol{E}} = \frac{\partial^2 \vec{\boldsymbol{E}}}{\partial z^2}$$

$$\frac{\partial^2 \vec{\boldsymbol{E}}}{\partial z^2} + \omega^2\, \mu_0\, \varepsilon_0\, \vec{\boldsymbol{E}} = 0 \qquad (8.67)$$

This is 2^{nd} order homogeneous differential equation. In order to find its solution, we need to find its characteristic or auxiliary equation. Let

$$\frac{\partial}{\partial z} = m$$

$$m^{\,2}\, \vec{\boldsymbol{E}} + \omega^2\, \mu_0\, \varepsilon_0\, \vec{\boldsymbol{E}} = 0 \qquad (8.68)$$

$$(m^{\,2} + \omega^2\, \mu_0\, \varepsilon_0)\, \vec{\boldsymbol{E}} = 0 \qquad (8.69)$$

$$(m^{\,2} + \omega^2\, \mu_0\, \varepsilon_0) = 0 \qquad (8.70)$$

This is the characteristic or auxiliary equation of the 2^{nd} order homogeneous differential equation. The roots of the auxiliary equation are imaginary

$$m = \pm\, j\omega\sqrt{\mu_0\, \varepsilon_0} \qquad (8.71)$$

As

$$m = \pm\,(\,\alpha + j\beta_0\,)$$

So

$$\alpha = 0$$

And

$$\beta_0 = \omega\sqrt{\mu_0\, \varepsilon_0}$$

The solution of the 2^{nd} order homogeneous differential equation is

$$\vec{\boldsymbol{E}} = \left(\, E_0\, e^{-j\beta_0 z} + E_b\, e^{\,j\beta_0 z}\,\right) \boldsymbol{a_x} \qquad (8.72)$$

Once again, the first component on the right hand side represents the component of the wave travelling along positive $z - axis$ and the second component represents the component of the wave travelling along negative $z-axis$. The first component is known as forward travelling wave and the second component is known as backward travelling wave. No component of the wave is travelling along negative $z-axis$, so we do ignore this component. When the reflection of the wave takes place due to change in the medium, then this component exists.

$$\vec{\boldsymbol{E}} = \left(E_0\, e^{-j\beta_0 z} \right) \boldsymbol{a_x} \tag{8.73}$$

This is the phasor value of the electric field intensity of the uniform plane wave, propagating in Free Space. Its instantaneous equation is given by

$$\boldsymbol{E} = E_0 \ \sin(\omega t - \beta_0 z)\boldsymbol{a_x}$$

8-9 Magnetic Field Intensity of TEM wave in Free Space

Consider the phasor value of the electric field intensity of the uniform plane wave, propagating in a lossless dielectric medium

$$\vec{\boldsymbol{E}} = \left(E_0\, e^{-j\beta_0 z} \right) \boldsymbol{a_x} \tag{8.74}$$

Lets us determine Magnetic Field Intensity of TEM waves in this medium with the help of

$$\boldsymbol{\nabla} \times \vec{\boldsymbol{E}} = -j\omega\, \mu_0 \vec{\boldsymbol{H}} \tag{8.75}$$

$$\vec{\boldsymbol{H}} = \frac{1}{-j\omega\, \mu_0} \left(\boldsymbol{\nabla} \times \vec{\boldsymbol{E}} \right) \tag{8.76}$$

$$\vec{\boldsymbol{H}} = \frac{1}{-j\omega\, \mu_0} \begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$\vec{\boldsymbol{H}} = \frac{1}{-j\omega\,\mu_0}\begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_0\, e^{-j\beta_0 z} & 0 & 0 \end{vmatrix}$$

$$\vec{\boldsymbol{H}} = \frac{-j\beta_0}{-j\omega\,\mu_0}\, E_0\, e^{-j\beta_0\, z}\, \boldsymbol{a_y}$$

$$\vec{\boldsymbol{H}} = \frac{-j\omega\sqrt{\mu_0\,\varepsilon_0}}{-j\omega\,\mu_0}\, E_0\, e^{-j\beta_0\, z}\, \boldsymbol{a_y}$$

$$\vec{\boldsymbol{H}} = \frac{1}{\sqrt{\frac{\mu_0}{\varepsilon_0}}}\, E_0\, e^{-j\beta_0\, z}\, \boldsymbol{a_y} \qquad (8.77)$$

Where

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

is the characteristic or intrinsic impedance of Free Space and is equal to 377Ω.

$$\vec{\boldsymbol{H}} = \frac{1}{\eta_0}\,(\,E_0\, e^{-j\beta_0 z}\,)\,\boldsymbol{a_y} \qquad (8.78)$$

This is the phasor value of the magnetic field intensity of the uniform plane wave, propagating in Free Space. Its instantaneous equation is given by

$$\boldsymbol{H} = \frac{E_0}{\eta_0}\,\sin(\omega t - \beta_0 z)\boldsymbol{a_y}$$

The average power density of the wave is given by

$$\boldsymbol{S_{ave}} = \frac{1}{2}\,Real\,(\,\vec{\boldsymbol{E}} \times \vec{\boldsymbol{H}^*}\,)$$

The maximum value of the speed of the wave takes place in free space that is given by

$$C = \frac{1}{\sqrt{\mu_0\ \varepsilon_0}} = 3 \times 10^8 \qquad m/sec \qquad (8.79)$$

The maximum wavelength is in Free Space as well

$$C = f\lambda_0 \qquad (8.80)$$

Example 8-4:

$\boldsymbol{E} = 94.25\ \sin(\omega t + 6z)\boldsymbol{a_x}$ is travelling in a medium characterized by $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$ find ω , $\boldsymbol{H}$, $v\ \ \lambda\ and$ average power density**.**

Answer

$$C = \frac{\omega}{\beta_0}$$

$$\omega = c\beta_0$$

$$\omega = 3 \times 10^8\ \times 6$$

$$\omega = 18 \times 10^8\ rad/sec$$

$$\lambda_0 = \frac{2\pi}{\beta_0}$$

$$\lambda_0 = \frac{2\pi}{6}$$

$$\lambda_0 = 1.047\ m$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$\eta_0 = \sqrt{\frac{4\ \pi \times 10^{-7}}{8.85 \times 10^{-12}}}$$

$$\eta_0 = 377\Omega$$

$$H = -\frac{E_0}{\eta_0}\ \sin(\omega t + \beta_0 z)\boldsymbol{a_y}$$

$$H =\ -0.25\ \sin(18 \times 10^8\ t + 6z)\boldsymbol{a_y}\ A/m$$

$$\boldsymbol{S_{av}} =\ -11.78\boldsymbol{a_z}\ w/m^2$$

Example 8-5:

$\boldsymbol{H} =\ 2 \times 10^{-3}\ \ \sin(2 \times 10^9\ t - \beta_0 y)\boldsymbol{a_x}$ is travelling in free space, find $\beta_0, \boldsymbol{E}$, $\lambda\ and$ average power density.

Answer

$$C = \frac{\omega}{\beta_0}$$

$$\beta_0 = \frac{\omega}{c}$$

$$\beta_0 = \frac{2 \times 10^9}{3 \times 10^8}$$

$$\beta_0 = 6.66$$

$$\lambda_0 = \frac{2\pi}{6.66}$$

$$\lambda_0 = 0.942\ m$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$\eta_0 = \sqrt{\frac{4\ \pi \times 10^{-7}}{8.85 \times 10^{-12}}}$$

$$\eta_0 = 377\Omega$$

$$\boldsymbol{E} = H_0\ \eta_0\ \sin(\omega t - \beta_0 y)\boldsymbol{a_z}$$

$$\boldsymbol{E} = 0.754 \sin(2 \times 10^9 t - 6.66z)\boldsymbol{a_z} \ A/m$$

$$\boldsymbol{S_{av}} = 0.754\boldsymbol{a_y} \ mw/m^2$$

8-10 Reflection of Uniform Plane Wave

When a uniform plane wave travels from one medium to another medium, a fraction of the energy of the wave is reflected back to the same medium and the remaining energy is transmitted to the second medium. The boundary between two media is defined as

Interface. We need to know the percentage of the energy that is lost due to reflection phenomenon. The reflection loss can be found in terms of reflection coefficient. In other words the reflection coefficient is a very important parameter that describes how much of an electromagnetic wave is reflected due to impedance discontinuity. If the medium is uniform then there will be no reflection at all. We shall consider three different interfaces in the forthcoming sections.

8-11 Lossless Dielectric-Lossless Dielectric Interface

Consider the interface between two lossless dielectric media that is located at $z = 0$ as shown in Figure 8-6. The permeability, permittivity, phase constant and intrinsic impedance of medium no 1 are represented by μ_0, ε_1, β_1 and η_1 respectively, while the permeability, permittivity, phase constant and intrinsic impedance of medium no 2 are represented by μ_0, ε_2, β_2 and η_2 respectively. The incident wave travels in medium 1 along $z - axis$ towards the interface. At the interface, a fraction of the amplitude of the incident wave is reflected back and the remaining energy is transmitted to the second medium.

Let us consider the waves in medium no 1.

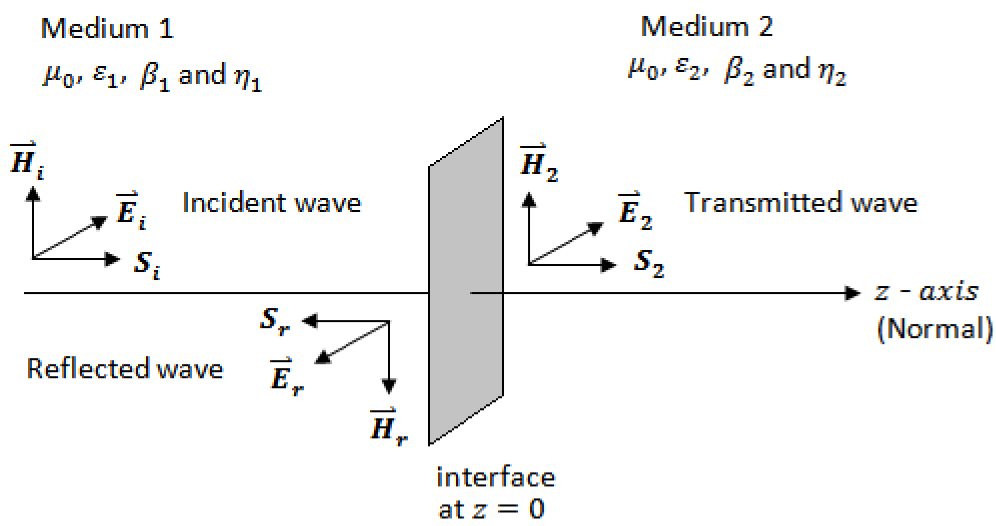


Figure 8-6: Reflection of the wave

Medium No 1:

Electric field intensity of the incident wave is given by

$$\vec{E}_i = \left(E_0 \, e^{-j\beta_1 z} \right) \boldsymbol{a_x} \tag{8.81}$$

Magnetic field intensity of the incident wave is given by

$$\vec{H}_i = \left(\frac{E_0}{\eta_1} \, e^{-j\beta_1 z} \right) \boldsymbol{a_y} \tag{8.82}$$

Average power density of the incident wave is given by

$$\boldsymbol{S_i} = \frac{1}{2} \, Real \, (\vec{E}_i \times \vec{H^*}_i)$$

Electric field intensity of the reflected wave is given by

$$\vec{E}_r = \left(\rho \, E_0 \, e^{j\beta_1 z} \right) \boldsymbol{a_x} \tag{8.83}$$

Where ρ is known as reflection coefficient.

Magnetic field intensity of the reflected wave is given by

$$\vec{\boldsymbol{H}}_{\boldsymbol{r}} = \left(-\frac{\rho E_0}{\eta_1}\ e^{j\beta_1 z}\right)\ \boldsymbol{a}_{\boldsymbol{y}} \tag{8.84}$$

Average power density of the reflected wave is given by

$$\boldsymbol{S}_{\boldsymbol{r}} = \frac{1}{2}\ Real\ (\ \vec{\boldsymbol{E}}_{\boldsymbol{r}}\ \times \vec{\boldsymbol{H}^*}_{\boldsymbol{r}}\ )$$

There are two waves having the same frequency in medium no 1, the incident wave and the reflected wave. Obviously the interference of these waves will take place, hence the total intensity in medium no 1 will be equal to the phasor sum of the incident and reflected wave.

Total electric field intensity in medium 1 is given by

$$\vec{\boldsymbol{E}}_{\boldsymbol{1}} = \vec{\boldsymbol{E}}_{\boldsymbol{i}} + \vec{\boldsymbol{E}}_{\boldsymbol{r}}$$

$$\vec{\boldsymbol{E}}_{\boldsymbol{1}} = \left(E_0\ e^{-j\beta_1 z} + \rho\ E_0\ e^{j\beta_1 z}\right)\boldsymbol{a}_{\boldsymbol{x}} \tag{8.85}$$

Total magnetic field intensity in medium 1 is given by

$$\vec{\boldsymbol{H}}_{\boldsymbol{1}} = \vec{\boldsymbol{H}}_{\boldsymbol{i}} + \vec{\boldsymbol{H}}_{\boldsymbol{r}}$$

$$\vec{\boldsymbol{H}}_{\boldsymbol{1}} = \left(\frac{E_0}{\eta_1}\ e^{-j\beta_1 z} - \rho\ \frac{E_0}{\eta_1}\ e^{j\beta_1 z}\right)\boldsymbol{a}_{\boldsymbol{y}} \tag{8.86}$$

Medium No 2:

Electric field intensity of the transmitted wave is given by

$$\vec{\boldsymbol{E}}_{\boldsymbol{2}} = \left(\ \tau E_0\ e^{-j\beta_2 z}\ \right)\ \boldsymbol{a}_{\boldsymbol{x}} \tag{8.87}$$

Where τ is known as transmission coefficient.

Magnetic field intensity of the transmitted wave is given by

$$\vec{\boldsymbol{H}}_{\boldsymbol{2}} = \left(\ \frac{\tau E_0}{\eta_2}\ e^{-j\beta_2 z}\ \right)\ \boldsymbol{a}_{\boldsymbol{y}} \tag{8.88}$$

Average power density of the transmitted wave is given by

$$S_2 = \frac{1}{2}\, Real\, (\overrightarrow{\boldsymbol{E}}_2 \times \overrightarrow{\boldsymbol{H}^*}_2)$$

We apply the boundary condition, which states that whenever a wave travels from one medium to another medium, its tangential component does not change however its normal component changes. As $\overrightarrow{\boldsymbol{E}}_1$ and $\overrightarrow{\boldsymbol{E}}_2$ are entirely tangential to the interface, therefore

$$\overrightarrow{\boldsymbol{E}}_1 = \overrightarrow{\boldsymbol{E}}_2$$

$$(\overrightarrow{\boldsymbol{E}}_1 = \overrightarrow{\boldsymbol{E}}_2) \text{ at } z = 0$$

$$1 + \rho = \tau \tag{8.89}$$

Similarly $\overrightarrow{\boldsymbol{H}}_1$ and $\overrightarrow{\boldsymbol{H}}_2$ are entirely tangential to the interface as well, therefore

$$\overrightarrow{\boldsymbol{H}}_1 = \overrightarrow{\boldsymbol{H}}_2$$

$$(\overrightarrow{\boldsymbol{H}}_1 = \overrightarrow{\boldsymbol{H}}_2) \text{ at } z = 0$$

$$\frac{1}{\eta_1} - \rho\,\frac{1}{\eta_1} = \frac{\tau}{\eta_2}$$

Putting the value of τ, we obtain

$$\frac{1}{\eta_1} - \rho\,\frac{1}{\eta_1} = \frac{1}{\eta_2} + \rho\,\frac{1}{\eta_2}$$

The reflection coefficient is given by

$$\rho = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \tag{8.90}$$

The transmission coefficient is given by

$$\tau = \frac{2\,\eta_2}{\eta_2 + \eta_1} \tag{8.91}$$

As the power is directly proportional to the square of the amplitude of electric field intensity, therefore power of the incident wave is

$$P_i = constant \times E_0^2$$

And power of the reflected wave is

$$P_r = constant \times \rho^2 E_0^2$$

Therefore ratio of the reflected power to the incident power is

$$\frac{P_r}{P_i} = \rho^2 \qquad (8.92)$$

***Example* 8 − 6**:

$\boldsymbol{E_i} = 377 \sin(10 \times 10^8\, t - 5z)\boldsymbol{a_x}$ is travelling from a medium characterized by $\mu = \mu_0$ and characterized by $\varepsilon_0 = 2.25$ to free space. Find $\boldsymbol{H_i}$, $\boldsymbol{E}_r$, $\boldsymbol{H}_r$, $\boldsymbol{E_2}$ and $\boldsymbol{H}_2$.

Solution

$$\eta_1 = 251.33\,\Omega\,,\ \eta_2 = 377\,\Omega$$

$$\beta_2 = \frac{\omega}{C}$$

$$\beta_2 = \frac{10 \times 10^8}{3 \times 10^8}$$

$$\beta_2 = 3.333$$

$$\rho = \frac{377 - 251.33}{377 + 251.33}$$

$$\rho = 0.2$$

$$\tau = \frac{377 \times 2}{377 + 251.33}$$

$$\tau = 1.2$$

$$\boldsymbol{E_i} = 377 \sin(10 \times 10^8\, t - 5z)\boldsymbol{a_x}$$

$$\boldsymbol{H_i} = 1.5 \sin(10 \times 10^8\, t - 5z)\boldsymbol{a_y}$$

$$\boldsymbol{S_i} = 282.75\boldsymbol{a_z}$$

$$\boldsymbol{E_r} = 377\rho \sin(10 \times 10^8\, t + 5z)\boldsymbol{a_x}$$

$$\boldsymbol{E_r} = 75.4 \sin(10 \times 10^8\, t + 5z)\boldsymbol{a_x}$$

$$\boldsymbol{H_r} = -1.5\rho \sin(10 \times 10^8\, t + 5z)\boldsymbol{a_y}$$

$$\boldsymbol{H_r} = -0.3 \sin(10 \times 10^8\, t + 5z)\boldsymbol{a_y}$$

$$\boldsymbol{S_r} = -11.31\boldsymbol{a_z}$$

$$\boldsymbol{E_2} = 377\tau \sin(10 \times 10^8\, t - 3.333z)\boldsymbol{a_x}$$

$$\boldsymbol{E_2} = 452.4 \sin(10 \times 10^8\, t - 3.333z)\boldsymbol{a_x}$$

$$\boldsymbol{H_2} = \frac{452.4}{377} \sin(10 \times 10^8\, t - 3.333z)\boldsymbol{a_y}$$

$$\boldsymbol{H_2} = 1.2 \sin(10 \times 10^8\, t - 3.333z)\boldsymbol{a_y}$$

$$\boldsymbol{S_2} = 271.44\boldsymbol{a_z}$$

8-12 Lossy Dielectric-Lossy Dielectric Interface

Consider the interface between two lossy dielectric media that is located at $z = 0$ as shown in Figure 8-7. The permeability, permittivity, propagation constant and intrinsic impedance of medium no 1 are represented by μ_0, ε_1, $\hat{\gamma}_1$and $\hat{\eta}_1$ respectively, while the permeability, permittivity, propagation constant and intrinsic impedance of medium no 2 are represented by μ_0, ε_2, $\hat{\gamma}_2$and $\hat{\eta}_2$ respectively.

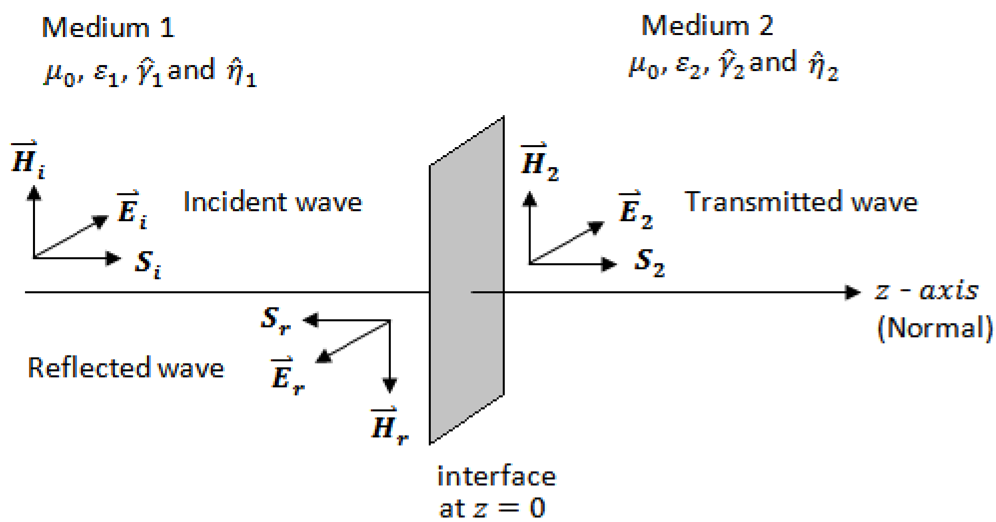


Figure 8-7: Reflection of the wave

The incident wave travels in medium 1 along $z - axis$ towards the interface. At the interface, a fraction of the amplitude of the incident wave is reflected back and the remaining energy is transmitted to the second medium.

Let us consider the waves in medium no 1.

Medium No 1:

Electric field intensity of the incident wave is given by

$$\vec{E}_i = \left(E_0 \, e^{-\hat{\gamma}_1 z} \right) \boldsymbol{a_x} \tag{8.93}$$

Magnetic field intensity of the incident wave is given by

$$\vec{H}_i = \left(\frac{E_0}{\hat{\eta}_1} \, e^{-\hat{\gamma}_1 z} \right) \boldsymbol{a_y} \tag{8.94}$$

Average power density of the incident wave is given by

$$\boldsymbol{S_i} = \frac{1}{2} \, Real \left(\vec{E}_i \times \vec{H^*}_i \right)$$

Electric field intensity of the reflected wave is given by

$$\vec{\boldsymbol{E}}_{\boldsymbol{r}} = \left(\rho\, E_0\, e^{\hat{\gamma}_1 z} \right) \boldsymbol{a_x} \tag{8.95}$$

Where ρ is known as reflection coefficient.

Magnetic field intensity of the reflected wave is given by

$$\vec{\boldsymbol{H}}_{\boldsymbol{r}} = \left(-\frac{\rho E_0}{\hat{\eta}_1}\, e^{\hat{\gamma}_1 z} \right) \boldsymbol{a_y} \tag{8.96}$$

Average power density of the reflected wave is given by

$$\boldsymbol{S_r} = \frac{1}{2}\, Real\, (\vec{\boldsymbol{E}}_{\boldsymbol{r}} \times \overrightarrow{\boldsymbol{H}^{*}}_{\boldsymbol{r}})$$

There are two waves having the same frequency in medium no 1, the incident wave and the reflected wave. Obviously the interference of these waves will take place, hence the total intensity in medium no 1 will be equal to the phasor sum of the incident and reflected wave.

Total electric field intensity in medium 1 is given by

$$\vec{\boldsymbol{E}}_{\boldsymbol{1}} = \vec{\boldsymbol{E}}_{\boldsymbol{i}} + \vec{\boldsymbol{E}}_{\boldsymbol{r}}$$

$$\vec{\boldsymbol{E}}_{\boldsymbol{1}} = \left(E_0\, e^{-\hat{\gamma}_1 z} + \rho\, E_0\, e^{\hat{\gamma}_1 z} \right) \boldsymbol{a_x} \tag{8.97}$$

Total magnetic field intensity in medium 1 is given by

$$\vec{\boldsymbol{H}}_{\boldsymbol{1}} = \vec{\boldsymbol{H}}_{\boldsymbol{i}} + \vec{\boldsymbol{H}}_{\boldsymbol{r}}$$

$$\vec{\boldsymbol{H}}_{\boldsymbol{1}} = \left(\frac{E_0}{\hat{\eta}_1}\, e^{-\hat{\gamma}_1 z} - \rho\, \frac{E_0}{\hat{\eta}_1}\, e^{\hat{\gamma}_1 z} \right) \boldsymbol{a_y} \tag{8.98}$$

Medium No 2:

Electric field intensity of the transmitted wave is given by

$$\vec{\boldsymbol{E}}_{\boldsymbol{2}} = \left(\tau E_0\, e^{-\hat{\gamma}_2 z} \right) \boldsymbol{a_x} \tag{8.99}$$

Where τ is known as transmission coefficient.

Magnetic field intensity of the transmitted wave is given by

$$\overrightarrow{\boldsymbol{H}}_{\boldsymbol{2}} = \left(\frac{\tau E_0}{\hat{\eta}_2}\, e^{-\hat{\gamma}_2 z} \right) \boldsymbol{a_y} \tag{8.100}$$

Average power density of the transmitted wave is given by

$$\boldsymbol{S_2} = \frac{1}{2}\, Real\, (\overrightarrow{\boldsymbol{E}}_{\boldsymbol{2}} \times \overrightarrow{\boldsymbol{H}^*}_{\boldsymbol{2}})$$

We apply the boundary condition, which states that whenever a wave travels from one medium to another medium, its tangential component does not change however its normal component changes. As $\overrightarrow{\boldsymbol{E}}_{\boldsymbol{1}}$ and $\overrightarrow{\boldsymbol{E}}_{\boldsymbol{2}}$ are entirely tangential to the interface, therefore

$$\overrightarrow{\boldsymbol{E}}_{\boldsymbol{1}} = \overrightarrow{\boldsymbol{E}}_{\boldsymbol{2}}$$

$$(\overrightarrow{\boldsymbol{E}}_{\boldsymbol{1}} = \overrightarrow{\boldsymbol{E}}_{\boldsymbol{2}}) \text{ at } z = 0$$

$$1 + \rho = \tau \tag{8.101}$$

Similarly $\overrightarrow{\boldsymbol{H}}_{\boldsymbol{1}}$ and $\overrightarrow{\boldsymbol{H}}_{\boldsymbol{2}}$ are entirely tangential to the interface as well, therefore

$$\overrightarrow{\boldsymbol{H}}_{\boldsymbol{1}} = \overrightarrow{\boldsymbol{H}}_{\boldsymbol{2}}$$

$$(\overrightarrow{\boldsymbol{H}}_{\boldsymbol{1}} = \overrightarrow{\boldsymbol{H}}_{\boldsymbol{2}}) \text{ at } z = 0$$

$$\frac{1}{\hat{\eta}_1} - \rho\,\frac{1}{\hat{\eta}_1} = \frac{\tau}{\hat{\eta}_2} \tag{8.102}$$

Putting the value of τ, we obtain

$$\frac{1}{\hat{\eta}_1} - \rho\,\frac{1}{\hat{\eta}_1} = \frac{1}{\hat{\eta}_2} + \rho\,\frac{1}{\hat{\eta}_2}$$

The reflection coefficient is given by

$$\rho = \frac{\hat{\eta}_2 - \hat{\eta}_1}{\hat{\eta}_2 + \hat{\eta}_2} \tag{8.103}$$

The transmission coefficient is given by

$$\tau = \frac{2\hat{\eta}_2}{\hat{\eta}_2 + \hat{\eta}_2} \tag{8.104}$$

8-13 Lossless Dielectric-Perfect Conductor Interface

Consider the interface between a lossless dielectric medium and a perfect conductor that is located at $z = 0$ as shown in Figure 8-8. The permeability, permittivity, phase constant and intrinsic impedance of medium no 1 are represented by μ_0, ε_1, β_1 and η_1 respectively, while the permeability, permittivity, propagation constant and intrinsic impedance of medium no 2 are represented by μ_0, ε_2, $\hat{\gamma}_2$ and $\hat{\eta}_2$ respectively. The incident wave travels in medium 1 along $z - axis$ towards the interface. At the interface, the amplitude of the incident wave is reflected back and no energy is transmitted to the second medium.

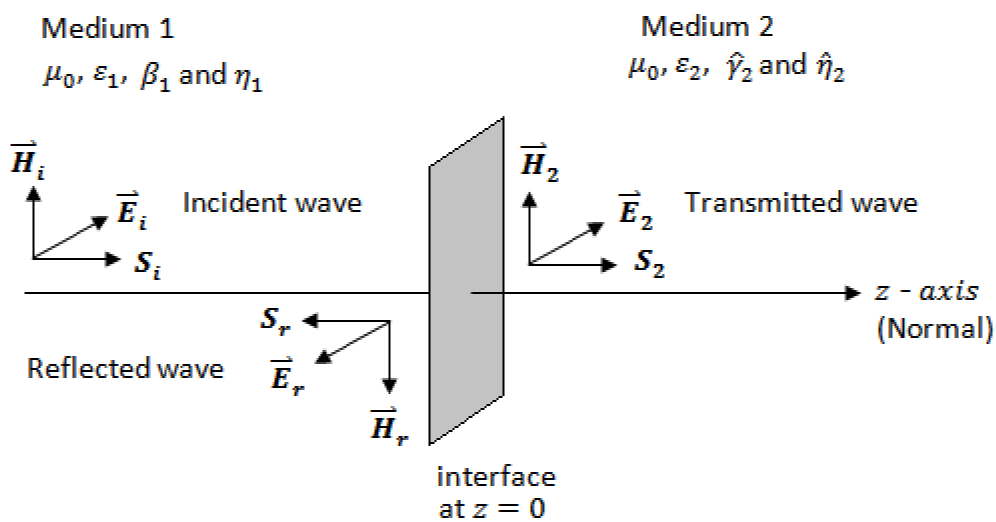


Figure 8-8: Reflection of the wave

Let us consider the waves in medium no 1.

Medium No 1:

Electric field intensity of the incident wave is given by

$$\vec{E}_i = \left(E_0 \, e^{-j\beta_1 z} \right) \boldsymbol{a}_x \tag{8.105}$$

Magnetic field intensity of the incident wave is given by

$$\vec{\boldsymbol{H}}_{\boldsymbol{i}} = \left(\frac{E_0}{\eta_1} \, e^{-j\beta_1 z} \right) \boldsymbol{a_y} \qquad (8.106)$$

Average power density of the incident wave is given by

$$\boldsymbol{S_i} = \frac{1}{2} \, Real \, (\vec{\boldsymbol{E}}_{\boldsymbol{i}} \times \vec{\boldsymbol{H^*}}_{\boldsymbol{i}})$$

Electric field intensity of the reflected wave is given by

$$\vec{\boldsymbol{E}}_{\boldsymbol{r}} = \left(\rho \, E_0 \, e^{j\beta_1 z} \right) \boldsymbol{a_x} \qquad (8.107)$$

Where ρ is known as reflection coefficient.

Magnetic field intensity of the reflected wave is given by

$$\vec{\boldsymbol{H}}_{\boldsymbol{r}} = \left(-\frac{\rho E_0}{\eta_1} \, e^{j\beta_1 z} \right) \boldsymbol{a_y} \qquad (8.108)$$

Average power density of the reflected wave is given by

$$\boldsymbol{S_r} = \frac{1}{2} \, Real \, (\vec{\boldsymbol{E}}_{\boldsymbol{r}} \times \vec{\boldsymbol{H^*}}_{\boldsymbol{r}})$$

There are two waves having the same frequency in medium no 1, the incident wave and the reflected wave. Obviously the interference of these waves will take place, hence the total intensity in medium no 1 will be equal to the phasor sum of the incident and reflected wave.

Total electric field intensity in medium 1 is given by

$$\vec{\boldsymbol{E}}_{\boldsymbol{1}} = \vec{\boldsymbol{E}}_{\boldsymbol{i}} + \vec{\boldsymbol{E}}_{\boldsymbol{r}}$$

$$\vec{\boldsymbol{E}}_{\boldsymbol{1}} = \left(E_0 \, e^{-j\beta_1 z} + \rho \, E_0 \, e^{j\beta_1 z} \right) \boldsymbol{a_x} \qquad (8.109)$$

Total magnetic field intensity in medium 1 is given by

$$\vec{\boldsymbol{H}}_{\boldsymbol{1}} = \vec{\boldsymbol{H}}_{\boldsymbol{i}} + \vec{\boldsymbol{H}}_{\boldsymbol{r}}$$

$$\vec{\boldsymbol{H}}_{\boldsymbol{1}} = \left(\frac{E_0}{\eta_1} \, e^{-j\beta_1 z} - \rho \, \frac{E_0}{\eta_1} \, e^{j\beta_1 z} \right) \boldsymbol{a_y} \qquad (8.110)$$

Medium No 2:

Electric field intensity of the transmitted wave is given by

$$\vec{\boldsymbol{E}}_{\mathbf{2}} = \left(\tau E_0 \, e^{-\hat{\gamma}_2 \, z} \right) \boldsymbol{a_x} \tag{8.111}$$

Where τ is known as transmission coefficient.

Magnetic field intensity of the transmitted wave is given by

$$\vec{\boldsymbol{H}}_{\mathbf{2}} = \left(\frac{\tau E_0}{\hat{\eta}_2} \, e^{-\hat{\gamma}_2 \, z} \right) \boldsymbol{a_y} \tag{8.112}$$

Average power density of the transmitted wave is given by

$$\boldsymbol{S}_{\mathbf{2}} = \frac{1}{2} \, Real \, (\vec{\boldsymbol{E}}_{\mathbf{2}} \times \overrightarrow{\boldsymbol{H}^{*}}_{\mathbf{2}})$$

We apply the boundary condition, which states that whenever a wave travels from one medium to another medium, its tangential component does not change however its normal component changes. As $\vec{\boldsymbol{E}}_{\mathbf{1}}$ and $\vec{\boldsymbol{E}}_{\mathbf{2}}$ are entirely tangential to the interface, therefore

$$\vec{\boldsymbol{E}}_{\mathbf{1}} = \vec{\boldsymbol{E}}_{\mathbf{2}}$$

$$(\vec{\boldsymbol{E}}_{\mathbf{1}} = \vec{\boldsymbol{E}}_{\mathbf{2}}) \text{ at } z = 0$$

$$1 + \rho = \tau$$

Similarly $\vec{\boldsymbol{H}}_{\mathbf{1}}$ and $\vec{\boldsymbol{H}}_{\mathbf{2}}$ are entirely tangential to the interface as well, therefore

$$\vec{\boldsymbol{H}}_{\mathbf{1}} = \vec{\boldsymbol{H}}_{\mathbf{2}}$$

$$(\vec{\boldsymbol{H}}_{\mathbf{1}} = \vec{\boldsymbol{H}}_{\mathbf{2}}) \text{ at } z = 0$$

$$\frac{1}{\eta_1} - \rho \, \frac{1}{\eta_1} = \frac{\tau}{\hat{\eta}_2} \tag{8.113}$$

Putting the value of τ, we obtain

$$\frac{1}{\eta_1} - \rho\,\frac{1}{\eta_1} = \frac{1}{\hat{\eta}_2} + \rho\,\frac{1}{\hat{\eta}_2}$$

The reflection coefficient is given by

$$\rho = \frac{\hat{\eta}_2 - \eta_1}{\hat{\eta}_2 + \eta_1} \qquad (8.114)$$

The transmission coefficient is given by

$$\tau = \frac{2\,\hat{\eta}_2}{\hat{\eta}_2 + \eta_1} \qquad (8.115)$$

As impedance of the perfect conductor is zero therefore $\rho = -1$ and $\tau = 0$, which means that total energy of the incident wave will be reflected back and no energy will be transmitted to the second medium.

References

1. *Electromagnetic Field Theory, By Hozorgu & Guru*
2. *Engineering Electromagnetics, 5th Edition by William H. Hayt, Jr.*

Dr Gulzar Ahmad is Associate Professor in the Department of Electrical Engineering, University of Engineering & Technology, Peshawar, Pakistan, where he has been a member of faculty since 1995. He is author of numerous publications in the field of Electrical Engineering and Microstip Patch Antenna. He is also the author of Basic Electrical Engineering book that was published in 2019.

Electromagnetic Field Theory is a book for the undergraduate students of the Department of Electrical Engineering. This book uses simple language and explains fundamental concepts with the help of diagrams. Important theoretical and mathematical results are given with the accompanying lengthy proofs, which I think is the main characteristic of the book. Solved numerical problems have been added to give the students the confidence in understanding the material presented. This book covers the topics of basic Electromagnetic Field Theory with the objective of learning and motivation. Easy explanation of topics and plenty of solved relevant examples is the principal features of this book.

ISBN 978-969-23460-0-9